Inter-Dealer Trading in Financial Markets*

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ABSTRACT

This paper compares alternative multi-stage inter-dealer mechanisms in financial markets with each other and with one-stage mechanisms traditionally analyzed in market microstructure. We consider three arrangements: a single uniform-price auction, a sequence of uniform-price auctions (sequential trading), and a limit-order book. With uninformative customer orders, sequential trading mechanisms dominate one-stage trading and limit-order books. The winning dealer in the first stage gets a larger share of the customer order and thus the customer is able to make the dealers compete more aggressively for the customer order. With informative customer orders, competition is negatively affected because the winning dealer attempts to use his information about the order flow to manipulate the perception of losing dealers. Sequential dealership markets break down when the customer order flow is too informative, while the limit-order book is robust and has higher revenues.

1 Introduction

Inter-dealer trading between dealers who act as market makers is a common feature of most major financial markets. With the exception of equity markets that deal primarily with relatively small order flows, in most dealership markets the outside order is filled by one dealer (often as a result of a negotiation process that is hardly anonymous) who then retrades with other dealers via an inter-dealer process. In equity markets such as the London Stock Exchange, inter-dealer trading constitutes some 40% of the total volume, and occurs mainly via an anonymous limit-order book although some inter-dealer trading occurs through direct negotiation between dealers.[1] In the market for U.S. Treasuries, two-thirds of the transactions are handled by inter-dealer brokerage firms such as Garban and Cantor-Fitzgerald while the remaining one third is done via direct dealings between the primary dealers. To improve market transparency, in 1990, the primary dealers and four inter-dealer brokers founded GovPX which acts as a disseminator of transaction price and volume information.[2] Hence, the actual trading between primary dealers takes place within each inter-dealer system but is reported via the GovPX system to financial institutions.[3] More recently, this idea of keeping competing inter-dealer systems but having a single reporting system like GovPX for transparency purposes has been implemented by the Bond Market Association for investment-grade bonds in response to the pressure from the SEC for greater transparency in the bond market.[4]

In the foreign exchange market, inter-dealer trading far exceeds public trades and accounts for about 85% of the trades.[5] Traditionally, inter-dealer trading on the foreign exchange market has been either by direct negotiation or brokered. Much of the inter-dealer trading via direct negotiation is sequential in nature (outside customer trades with dealer 1 who trades with dealer 2 who trades with dealer 3 and so on). This quick inter-dealer trading between a sequence of dealers is often referred to as "*hot potato*" trading. Today, 90% of this direct inter-dealer trading is done via the Reuters D2000-1 system. The Reuters system allows for

[1] See Hansch, Haik and Viswanathan (1998), Naik and Yadav (1997), and Reiss and Werner (1997).

[2] Cantor-Fitzgerald does not participate in GovPX because it has an agreement with Dow-Jones Telerate.

[3] The GovPX web site http://www.govpx.com provides useful information on the history of GovPX.

[4] See the press release from the Bond Market Association at web site http://www.bondmarkets.com or GovPX news at http://www.govpx.com. In an appearance before a House subcommittee on September 29, 1998, SEC Chairman Arthur Levitt pushed for greater transparency in bond trading.

[5] See Lyons (1995) for more on this.

bilateral electronic conversations in which one dealer asks another for a quote. The provided quote is then quickly (within seconds) accepted or rejected by the originating dealer.[6] The second method of inter-dealer trading is brokered trading. This brokered trading is today executed via one of two electronic limit-order book systems, Reuters (the Reuters Dealing 2002 system) and Electronic Brokerage System (EBS Spot Dealing System). The Reuters Dealing 2002 system was launched in 1992 and is discussed extensively by Goodhart, Ito and Payne (1996). The EBS system was started to compete against the Reuters system and claims average volumes in excess of $80 billion a day.[7] Hence, in most financial markets, inter-dealer trading is an integral part of market design, particularly for institutional markets that deal with large orders. At the same time, the exact structure of inter-dealer trading is undergoing significant change.

In our view, the literature on market microstructure does not explain how the inter-dealer process should differ according to the underlying trading environment.[8] While the seminal work of Ho and Stoll (1983) suggests that risk sharing is a strong motive behind inter-dealer trading, it is not entirely clear why risk sharing needs could not be met optimally by direct customer trading with several dealers. Even if inter-dealer trading is desirable, it is unclear as to which method of inter-dealer trading is more appropriate. Our view is echoed by Lyons (1996a) in his discussion of empirical results on the foreign exchange market where he states that *"a microstructural understanding of this market requires a much richer multiple-dealer theory than now exists (see e.g. Ho and Stoll 1983)"*.

In this paper, we provide models of inter-dealer trading that are based both on single-price procedures with multiple rounds of trading (reflecting the traditional voice-broking methods of inter-dealer trading) and on multiple-price procedures like limit-order books (reflecting the recent move towards electronic books in the foreign exchange market). Also, we ask how

[6]This information is provided in Evans and Lyons (1999) that uses the Reuters-2001 dataset.

[7]The EBS partnership is formed by a consortium of banks that are the leading market makers in the foreign exchange market. One key advantage of the EBS spot dealing system is that it links automatically with FXNET, a separate limited partnership of banks, that provides for automated netting and hence reduces settlement risk. EBS's web site (http://www.ebsp.com) provides more details. The Reuters web site (http://www.reuters.com) provides details on the Reuters Dealing 2002 system but very little about the older Reuters D2000-1 system.

[8]In the canonical models of Glosten and Milgrom (1985) and Kyle (1985), the outside order is taken completely by one dealer and no retrading occurs.

the strategic behavior of dealers and the execution prices for customer orders differ with the exact structure of the inter-dealer market (single-price auction versus limit-order book) and with the motivation of customer trades (inventory versus information). We believe that allowing for realistic features such as inter-dealer trade and "hot potato" trading is crucially important for improved understanding of dealership markets such as the foreign exchange market and the NASDAQ market.

Initially, we compare customer welfare between two-stage trading mechanisms that involve inter-dealer trading after a customer-dealer trade on the one hand, and the one-shot settings traditionally analyzed in the literature on the other. We identify two main advantages of two-step processes that are absent in one-shot trading environments.

First, dealers' opportunity to retrade produces an asymmetric quantity distribution — there is a "quantity bias" toward the dealer who fills the customer order in the first stage (i.e., he will end up owning more than his "fair" share). This provides strong incentives for the dealers to engage in more heightened competition for the customer order. In fact, because the dealers compete with flatter (i.e., more elastic) demand curves, customer welfare is generally improved relative to a one-shot mechanism.

Second, starting with the work of Wilson (1979), it is well known that single-price divisible good auction mechanisms are often plagued with a "demand reduction" problem.[9] That is, uniform-price auctions have equilibria in which prices deviate substantially from the economic value. The use of two-stage mechanisms can alleviate the problem to a large degree because customers do not split orders in the first stage, so the scope of strategic bidding on the part of dealers is severely limited. This implies that, because the dealers have to bid for either the whole customer order or none of it, they cannot set artificially low prices, as they could if order-splitting were permitted. The ability of two-stage procedures to elicit greater bidder competition provides the seller with a potentially powerful weapon in coping with strategic bidding in single-price auction mechanisms. This idea, we believe, is worth some emphasis and is useful in understanding multi-stage selling mechanisms outside the dealership setting.

To gain a deeper understanding of the nature of inter-dealer trading, we extend the analysis to the case when the dealers rely on a limit-order book at the second-stage. There

[9]See, e.g., Ausubel and Cramton (1996) and Back and Zender (1993).

are some notable differences between inter-dealer trading at one price (dealership) and inter-dealer trading at multiple prices (a limit-order book). In the latter setting, the cost of two-stage mechanisms is that a flatter demand curve translates into a smaller amount of surplus that the customer extracts from the dealers. Furthermore, in contrast to the dealership market, the benefit to a limit-order book market *decreases* with customer order size. This is related to the fact that bid-reduction (i.e., departure from pricing according to marginal valuation) is the greatest at large quantities for a single-price auction, and it is the greatest at small quantities for a limit-order book.[10] Therefore, in the case of limit-order book trading, the improvement of a two-stage versus a one-shot mechanism is mostly for relatively small customer orders.

We investigate a sequential auction procedure with multiple rounds of unit-auctions. This procedure involves the winning dealer (in any round) keeping some fraction of the object and selling the remaining via a unit-auction to one of the remaining dealers in that round. The procedure is in the spirit of the "voice-broking" market where a customer sells to dealer 1 who then sells to dealer 2 and so on. It generalizes our two-stage, single-price procedure and involves sequential trading among the dealers. We show that in such a multi-stage procedure, market liquidity falls as the auction progresses and in each successive round the winning dealer in that round keeps a larger fraction of the order flow for himself. Further, we show that the seller is better off with more rounds of auctions. These results rationalize the use of sequential trading mechanisms and provide an explanation for the phenomenon of "hot potato" trading.

We also extend the analysis to environments where the customer order flow contains payoff-relevant information.[11] In this situation, absent reporting of trades, the information asymmetry between the customer and the market makers imposes a cost on inter-dealer trading that adversely affects the attainment of efficient risk sharing. In particular, the winning dealer attempts to use his knowledge of the customer order flow to manipulate the beliefs of the losing dealers via the inter-dealer market. As a result, the degree of price

[10] These differences in market makers' trading strategies across market structures are emphasized in Viswanathan and Wang (1997). The hybrid market structure considered there is not a two-stage mechanism, but rather a concurrent setup which routes orders to different marketplaces using a size criterion.

[11] It seems that inventory and payoff-relevant information are important aspects of foreign exchange trading.

competition is uniformly lower compared to the case of noninformative order flows. In fact, with a severe enough adverse selection problem, (linear strategy) equilibrium does not exist in a multi-stage dealership (or auction) market. In contrast, inter-dealer trading with a limit-order book is less sensitive to private customer information and does not break down. This indicates that too much information in the customer order flow is a problem for sequential dealership trading and thus a limit-order book is preferable. Thus, the electronic inter-dealer system is more robust to market breakdowns than traditional voice-broking.

The results on informative order flow in dealership markets are consistent with the findings of Naik, Neuberger, and Viswanathan (1999) and Lyons (1997), although our setup is very different from previous work.[12] Vogler (1997) considers the case when dealers have the same pretrading inventory and concludes that inter-dealer trading always dominates the one-shot uniform-price auction setup. Potential costs to two-stage trading in our model are absent from Vogler's model because his is not a model of private information and because he assumes homogeneous inventories across the dealers. None of the above papers considers the limit-order book as a possible mode of inter-dealer trading or considers sequential auction procedures.[13] Importantly, our modeling of the limit-order book allows dealers to use limit-orders of arbitrary sizes and optimal bidding in the first round for the customer trade.

This paper is also related to the literature on market mechanisms that allow the dealers to bid for a fraction of the outside order. This divisibility of the order flow has been emphasized by a number of authors including Bias, Martimort, and Rochet (1997), Viswanathan and Wang (1997) and Röell (1997). Viswanathan and Wang (1997) characterize the equilibria of a dealership market (a single-price setup) and a limit-order book (a multi-price setup) when the risk averse dealers compete for the customer order via downward sloping demand curves. Röell (1997) provides related results when the order flow is drawn from the exponential distribution. Bias, Martimort, and Rochet (1997) provide a related characterization for the limit-order book when the marginal valuation curve is downward sloping due to informational

[12]See also Perraudin and Vitale (1996) for a model with information flows and risk neutral dealers.

[13]Werner (1997) presents a double auction model of inter-dealer trading with initially identical dealers. Following the extensive literature on double auctions with unit demands, all dealers in Werner (1997) submit limit orders to buy or sell a fixed amount in the second stage. Inter-dealer trading is also taken as given in Lyons (1996b) where the focus is on the *transparency* of inter-dealer trades.

reasons.[14] While these papers provide characterizations of equilibria when the customer trades with all the dealers simultaneously, they do not consider alternative mechanisms in which inter-dealer retrading is possible.

The rest of the paper is planned as follows. In Section 2 we present a model of two-stage trading in a dealership setting when the *ex ante* dealer inventories are known and when the customer order flow is noninformative. The basic intuitions of the model are laid out in this simple context by comparing customer welfare between two-stage and one-stage trading environments. Section 3 analyzes a sequential auction procedure emphasized above. The important case of limit-order book trading is taken up in Section 4. Section 5 deals with the situation when customer trades are informative. There we also compare the customer's expected revenues under the different trading mechanisms studied in the paper. Concluding remarks are offered in Section 6.

2 Inter-Dealer Trading with a Single-Price Mechanism

2.1 The Model

There are $N > 2$ risk averse dealers (market makers or liquidity providers) in the game. Each of the dealers can potentially fill a sell order from a risk neutral outside customer[15] of size $\tilde{z}$, which is distributed over the unit interval $[0, 1]$. The dealers act to maximize a mean-variance derived utility of profit with the risk aversion parameter, ρ.[16] A typical dealer, generically referred to as dealer k ($k = 1, 2, ..., N$), is endowed with an *ex ante* inventory of $\tilde{I}_k$, which is drawn from some commonly known distribution that has the support, $[\underline{I}, \bar{I}]$. We denote the average (per dealer) initial inventory as $\tilde{Q} \equiv (1/N) \sum_{k=1}^{N} \tilde{I}_k$.[17] The underlying asset value,

[14] These papers generalize earlier work by Glosten (1989) on the monopoly specialist and by Glosten (1994) on the competitive limit-order book. See Viswanathan and Wang (1997) for the relationship between these 3 papers.

[15] Customer buys are analyzed analogously.

[16] With single-price inter-dealer trading (modeled as a uniform-price auction), this quadratic objective function may be derived from the standard combination of constant absolute risk aversion utility functions and normally distributed random variables. However, in the case of limit-order book trading (akin to a discriminatory auction), these assumptions do not imply a quadratic objective function in the dynamic optimization problem.

[17] Since $\tilde{I}_k$ is private information known only to dealer k, $\tilde{Q}$ will be a random variable from any single dealer's standpoint. However, in solving the equilibrium in a limit-order book, we require that Q be known by all dealers.

$\tilde{v}$, is normally distributed as $N(\bar{v}, \tau_v^{-1})$, and $\tilde{v}$ is independent of the *ex ante* inventories, $\tilde{I}_k$.

To isolate the difference between two-stage trading and one-shot trading models, we initially assume that the *ex ante* dealer inventories are common knowledge among all dealers. Main conclusions of the section are not affected when this assumption is relaxed (see Section 2.4). Furthermore, we assume that the customer order does not carry information about the fundamental value of the traded asset. The consequences of relaxing this assumption will be evaluated in Section 5.

As benchmark, we will consider a one-shot trading setup in which the dealers submit demand schedules to compete for fractions of the customer order which is split among the N dealers. The focus of this section, however, is two-stage mechanisms in which the outside order is first filled *in its entirety* by a single dealer before it is divided among all dealers via an inter-dealer trading process. Order splitting during customer-dealer trading is disallowed.[18] Partly because most dealership markets are not anonymous, the customer is expected to sell to one dealer instead of splitting up the order among several dealers. As such, competition among the market makers in the first round is similar to the bidding for an indivisible good.

The dealer who wins the customer order will be referred to as dealer W, while any dealer who loses the customer order is referred to as dealer L ($\forall L = 1, 2, ..., N,\ L \neq W$).[19] After the customer-dealer trading, the customer order size and transaction price become known to dealer W, but *not* to the other dealers.[20] Consequently, during inter-dealer trading, dealer W can condition his trading strategy on z, while dealer L cannot. Inter-dealer trading among

[18] This "all-or-nothing" assumption is adopted in part to avoid the difficulty of choosing among the multiple equilibria that result from analyzing a share auction. We point out that the main conclusions of the paper regarding the superiority of two-stage mechanism over one-stage mechanism are not driven by this assumption. As subsequent analysis will make clear, what is important is the observation that the winning dealer in the initial customer-dealer trading keeps a disproportionately large share in subsequent inter-dealer trading.

[19] Throughout the paper, the subscript $_k$ denotes quantities for a typical dealer k, whereas the superscript ^{W} (or ^{L}) denotes quantities for any specific dealer when he is identified as the winning (or losing) dealer in the first round.

[20] We point out that the impact of disclosure is particularly important if the second stage is run as a limit-order book, since a key aspect of a book is its inability to condition on quantity. If inter-dealer trading takes place in a dealership setting, however, the issue of disclosure is not expected to significantly alter the equilibrium outcome. In this paper, we do not pursue issues related to disclosure but rather focus on the comparison of various modes of inter-dealer trading. See Naik, Neuberger, and Viswanathan (1999) for a paper that focuses on disclosure and customer welfare.

all N dealers starts shortly after the first round.[21] The efficacy of such two-stage trading processes will be compared to one-shot trading in which the customer can directly trade with multiple market makers.

It is convenient to visualize inter-dealer trading as involving three steps. First, dealer W hands over his entire holding of the asset, $I^W + z$, to an auctioneer (the inter-dealer broker). Then, the auctioneer solicits bids from all dealers (including dealer W) in the form of combinations of price and quantity. Further, we restrict the analysis to considering demand schedules that are continuously differentiable and downward sloping. A typical dealer k's trading strategy, as a function of the equilibrium price and possibly his own pretrading net position, is the quantity that is awarded to him by the auctioneer, x_k.[22] The equilibrium price of a single-price inter-dealer trading, $\tilde{p}_2$, is determined by equating demand and supply. After the auctioneer collects payment from all winning bids (those with price levels at or above the equilibrium price), the total proceeds are then returned to dealer W and are his to keep. Note that, at the conclusion of the inter-dealer trading, dealer W's net holding is x^W, while dealer L's net position is $I^L + x^L$. In other words, x^W denotes dealer W's final allocation, while x^L is dealer L's trade quantity. Results in the paper need to be interpreted with this convention in mind.

In the two-stage game, the winning dealer submits a supply curve during inter-dealer trading. An alternative is for the winning dealer to use the quantity choice as a strategy. However, with two stages of trading and no adverse selection problem, we can show that it is an inferior mechanism for the winning dealers. The case of winning dealers making *sequential* quantity choices is analyzed in Section 3.

A distinct feature of the dealership market is that most transactions take place at a single price. Therefore, dealer k's profit during inter-dealer trading is:

$$\tilde{\pi}_k^W = \tilde{v}\tilde{x}_k^W + \tilde{p}_2(I_k + \tilde{z} - \tilde{x}_k^W), \text{ if } k \text{ wins the customer order,}$$

[21]In a dynamic setting, dealer W would face the tradeoff between waiting for the next customer buy to arrive and initiating trading with other dealers right away. By focusing on a process where inter-dealer trading occurs soon after the customer order is filled by one particular dealer, we are essentially studying markets where the need to lay off the risk associated with an unbalanced portfolio is very significant.

[22]It is often convenient to express the bidding strategies as the inverse demand schedule, i.e., price as a function of quantity and inventory. To save notation, arguments to the demand schedules are often suppressed.

$$\tilde{\pi}_k^L = \tilde{v}(I_k + \tilde{x}_k^L) - \tilde{p}_2 \tilde{x}_k^L. \qquad \text{if } k \text{ does not win the customer order,}$$

Important differences between the dealership setup (which is a single-price mechanism) and the case when inter-dealer trading occurs via a limit-order book (which is a multi-price mechanism) will be the subject matter of Section 4.

For ease of presentation,[23] we maintain the following restrictions throughout the paper:

$$\begin{aligned} I_k &\ll \frac{\bar{v}}{\rho\tau_v^{-1}}, \ \forall k, \\ \frac{1}{N} &\ll \frac{\bar{v}}{\rho\tau_v^{-1}}. \end{aligned}$$

With tractability in mind, we restrict our analysis to equilibria of the model that are characterized by linear trading strategies. Given the symmetry of the problem, we will search for equilibria in which the strategies of the nonwinning dealers ($\forall L \neq W$) take the same functional form.

2.2 Benchmark: A One-Shot Dealer Market

A one-shot model where the customer can trade directly with N competing market makers serves as a benchmark for comparison with the two-stage trading model just introduced. The following is a necessary condition for the equilibrium strategies in a single-price dealership market:[24]

$$\sum_{j \neq i}^{N} \frac{\partial x_j(p, I_j)}{\partial p} = -\frac{x_i(p)}{\bar{v} - p - \rho\tau_v^{-1}[I_i + x_i(p, I_i)]}.$$

A unique symmetric, linear solution to the above equation is the following:

$$x_k(p, I_k) = \gamma_d(\bar{v} - \rho\tau_v^{-1} I_k - p), \quad \forall k = 1, 2, ..., N,$$

where

$$\gamma_d = \frac{N-2}{(N-1)\rho\tau_v^{-1}}. \tag{1}$$

[23]We make these assumptions in order to restrict the discussion to just one side of the market, i.e., a customer sells and the market makers buy. Relaxing these parameter restrictions does not affect the conclusions of the paper in any substantive way.

[24]See Viswanathan and Wang (1997) for a derivation.

The equilibrium price and allocations are:

$$\tilde{p}_d = \bar{v} - \rho\tau_v^{-1}\tilde{Q} - \frac{(N-1)\rho\tau_v^{-1}}{N-2}\frac{\tilde{z}}{N}, \tag{2}$$
$$\tilde{x}_k = \frac{\tilde{z}}{N} + \frac{N-2}{N-1}(\tilde{Q} - I_k).$$

The customer's total revenue is

$$\tilde{R}_d = \tilde{z}\tilde{p}_d.$$

At the end of trading, the net inventory positions are as follows:

$$I_k + \tilde{x}_k = \frac{\tilde{z}}{N} + \frac{N-2}{N-1}\tilde{Q} + \frac{I_k}{N-1}.$$

In this one-shot trading game, the customer order itself is evenly divided among the N bidders. However, none of the dealers is able to hedge the quantity risk associated with his *ex ante* inventory completely.

2.3 Equilibrium Analysis of the Two-Stage Trading Model

The two-stage trading model is solved using backward induction. Under a dealership structure, the second-stage (i.e., inter-dealer competition) can be modeled as a single-price divisible good auction. The first round bidding between the market makers for the customer order can be viewed as a unit demand auction with private bidder valuation. With publicly known dealer inventories, the private values are common knowledge and the outcome of the first stage bidding is straightforward.

2.3.1 The Second Round: Inter-Dealer Trading

When the inter-dealer trading results in all trades clearing at a single price, the dealers' equilibrium trading strategies are provided below. The solution in Proposition 2.1 is quite robust in the sense that it is independent of the distribution of customer order size, $\tilde{z}$, or the dealer inventory, $\tilde{I}_k$.

Proposition 2.1 If inter-dealer trading occurs at a single price, it has a unique linear strategy equilibrium characterized by the following trading strategies:

$$x^W(p, z, I^W) = \gamma_2(\bar{v} - p) + \frac{I^W + z}{N-1}, \tag{3}$$

$$x^L(p, I^L) \;=\; \gamma_2(\bar{v} - p) - \frac{N-2}{N-1} I^L, \quad \forall L \neq W, \tag{4}$$

where the price elasticity of demand, γ_2, is given by:

$$\gamma_2 \;=\; \frac{N-2}{(N-1)\rho\tau_v^{-1}}. \tag{5}$$

Proof: See Appendix A. ∎

Using the above equilibrium strategies, it is straightforward to show:

$$\tilde{p}_2 \;=\; \bar{v} - \rho\tau_v^{-1}(\tilde{Q} + \frac{\tilde{z}}{N}), \tag{6}$$

$$\tilde{x}^W \;=\; \frac{N-2}{N-1}(\tilde{Q} + \frac{\tilde{z}}{N} + \frac{I^W + \tilde{z}}{N-2}), \tag{7}$$

$$\tilde{x}^L \;=\; \frac{N-2}{N-1}(\tilde{Q} + \frac{\tilde{z}}{N} - I^L),\ \forall L \neq W. \tag{8}$$

Thus, the dealers' net positions at the end of two rounds of trading will be:

$$\begin{aligned} \tilde{x}^W &= \frac{2\tilde{z}}{N} + \frac{N-2}{N-1}\tilde{Q} + \frac{I^W}{N-1}, \\ I^L + \tilde{x}^L &= \frac{N-2}{N-1}\frac{\tilde{z}}{N} + \frac{N-2}{N-1}\tilde{Q} + \frac{I^L}{N-1}. \end{aligned}$$

Notice that, at the conclusion of inter-dealer trading, the customer order is split unevenly among the dealers, with dealer W retaining an above-average fraction $2/N$, and every other dealer retaining a below-average fraction $(N-2)/[N(N-1)]$, of the original customer order $\tilde{z}$. This is a key difference between one-shot trading and two-stage trading.

Using the last two relations, we can write dealer k's ($\forall k = 1, 2, ..., N$) second-stage (certainty equivalent) utility, depending on whether or not he gets to fill the customer order in the first round:

$$\begin{aligned} \tilde{U}_k^W &= \bar{v}\tilde{x}_k^W - \frac{\rho\tau_v^{-1}}{2}(\tilde{x}_k^W)^2 + \tilde{p}_2(I_k + \tilde{z} - \tilde{x}_k^W) \\ &= \bar{v}[\frac{N-2}{N-1}(\tilde{Q} + \frac{\tilde{z}}{N} + \frac{I_k + \tilde{z}}{N-2})] - \frac{\rho\tau_v^{-1}}{2}[\frac{N-2}{N-1}(\tilde{Q} + \frac{\tilde{z}}{N} + \frac{I_k + \tilde{z}}{N-2})]^2 \\ &\quad + [\bar{v} - \rho\tau_v^{-1}(\tilde{Q} + \frac{\tilde{z}}{N})][I_k + \tilde{z} - \frac{N-2}{N-1}(\tilde{Q} + \frac{\tilde{z}}{N} + \frac{I_k + \tilde{z}}{N-2})], \\ \tilde{U}_k^L &= \bar{v}(I_k + \tilde{x}_k^L) - \frac{\rho\tau_v^{-1}}{2}(I_k + \tilde{x}_k^L)^2 - \tilde{p}_2\tilde{x}_k^L \end{aligned}$$

$$= \bar{v}[I_k + \frac{N-2}{N-1}(\tilde{Q} + \frac{\tilde{z}}{N} - I_k)] - \frac{\rho\tau_v^{-1}}{2}[I_k + \frac{N-2}{N-1}(\tilde{Q} + \frac{\tilde{z}}{N} - I_k)]^2$$
$$-[\bar{v} - \rho\tau_v^{-1}(\tilde{Q} + \frac{\tilde{z}}{N})][\frac{N-2}{N-1}(\tilde{Q} + \frac{\tilde{z}}{N} - I_k)].$$

From the above it is straightforward to compute the utility difference between the winning dealer and the losing dealers which is useful in analyzing the first stage of customer-dealer trading.

Corollary 2.1 If inter-dealer trading occurs at a single price, the difference in dealer k's second-stage (certainty equivalent) utility between winning and losing the customer order, $\tilde{z}$, is:

$$\tilde{U}_k^W - \tilde{U}_k^L = \tilde{z}\{\bar{v} - \frac{\rho\tau_v^{-1}}{(N-1)^2}[I_k + N(N-2)\tilde{Q} + (N - \frac{3}{2})\tilde{z}]\}, \quad \forall k = 1, 2, ..., N. \quad (9)$$

2.3.2 The First Round: Customer-Dealer Trading

For any given customer order $\tilde{z}$, Corollary 2.1 reveals that dealer k's incentives for filling the customer order are based on the following private value function:[25]

$$\tilde{V}_k = \tilde{U}_k^W - \tilde{U}_k^L, \quad (10)$$

which is strictly decreasing in his own inventory level, I_k. Thus, without loss of generality, we can index the dealers in ascending order of their inventory positions, i.e., $I_1 < I_2 < ... < I_N$.

When dealer inventory is public knowledge among all market participants, first stage trading is equivalent to bargaining under complete information.[26] The dealer with the lowest *ex ante* inventory, dealer 1, wins the customer order, and he pays an amount equal to the reservation price of the dealer with the second-lowest inventory, dealer 2.

Proposition 2.2 Assume that dealer inventories are publicly known. If inter-dealer trading occurs at a single price, the customer receives the following rev-

[25]Due to the two-stage nature of the model, it is *not* always true that one could directly work with the certainty equivalent utility in computing the dealers' optimal trading strategy in the first round. See the Lemma in Appendix B for a proof of the validity of the certainty equivalent approach in our model.

[26]We assume that the market makers can make "take-it-or-leave-it" offers to the customer.

enue in the first round of trading:

$$\begin{aligned}\tilde{R}_1 &= \tilde{z}\{\bar{v} - \frac{\rho\tau_v^{-1}}{(N-1)^2}[I_2 + N(N-2)Q + (N - \frac{3}{2})\tilde{z}]\} \\ &\equiv \tilde{z}\tilde{p}_1, \end{aligned} \tag{11}$$

where I_2 is the second-smallest inventory among all the dealers.

PROOF: From Eq.s (9) and (10), it is clear that, for any given $\tilde{z}$, we have $\tilde{V}_1 > \tilde{V}_2 > ... > \tilde{V}_N$.

Although the dealers do not know in advance the size of the incoming customer order, $\tilde{z}$, they can bid by submitting a series of quantity-payment pairs, i.e., a demand schedule that states at each potential customer order size what the total payment to the customer, B_k, will be. In particular, the following is a set of (weakly) dominant bidding strategies (as a function of the customer order size, $\tilde{z}$):

$$\begin{aligned} B_1(z) &= V_2(z) + \epsilon, \\ B_k(z) &= V_k(z), \quad \forall k \neq 1. \end{aligned}$$

Regardless of the actual customer order sizes, the outcome is always that dealer 1 wins and pays the customer dealer 2's reservation price of $\tilde{V}_2$ (plus a small positive amount, ϵ). ∎

By comparing Eq. (11) and (2), we have:

$$\tilde{p}_1 - \tilde{p}_d = \frac{\rho\tau_v^{-1}}{(N-1)^2}[Q - I_2 + \frac{N^2-2}{2N(N-2)}\tilde{z}]. \tag{12}$$

Therefore, we have the following result.

Corollary 2.2 Assume that dealer inventories are publicly known. Relative to the equilibrium price in a one-shot game, $\tilde{p}_d$, the equilibrium price in the first round of the two-stage game, $\tilde{p}_1$, has two properties: (i) $\tilde{p}_1$ as a function of the customer order size $\tilde{z}$ is always flatter (i.e., more price-elastic) than $\tilde{p}_d$; (ii) $\tilde{p}_1$ has a higher intercept (i.e., small-quantity quote) than $\tilde{p}_d$ if and only if $Q > I_2$.

The intuition for the above result is as follows. We have shown that inter-dealer trading results in the winner in the first round retaining an above-average share of the customer order at the conclusion of inter-dealer trading. Given the quantity bias toward the winning dealer, the dealers compete more intensely for the customer order relative to the degree of competition in a one-shot model. The enhanced dealer competition (henceforth referred to as the "competition effect") is a net benefit to two-stage trading, and the size of this benefit increases as the customer order becomes larger.

When dealer inventory is publicly known, the first round of trading resembles a bargaining game. The winning dealer is able to extract a surplus from the customer which is equal to the difference between the private values of the dealers with the lowest and the second-lowest inventories. The extent of such surplus extraction, relative to the bidding profits in a one-shot model, is measured by the difference between the average *ex ante* dealer inventory and the second-lowest inventory. Depending on whether this difference is positive or negative, two-stage trading will provide the customer with a markup or markdown.

Since the zero-quantity pricing effect does not change as customer order sizes change, while the competition effect is proportional to the order size, the competition effect dominates at large order sizes. Therefore, the two-stage dealership market generally provides better execution than its one-shot counterpart for relatively large-sized order flows. See Figure 1 for an illustration.

In empirical work, Naik and Yadav (1997) and Reiss and Werner (1997) find that the inter-dealer spread is tighter than the spread on customer-dealer trades. This agrees with the following prediction of our model.

Corollary 2.3 Suppose dealer inventories are relatively homogeneous in size. For any given customer order, $\tilde{z}$, the bid-ask spread during inter-dealer trading is smaller than that during customer-dealer trading.

PROOF: From Eq.s (6) and (11), it is easy to see that $\tilde{p}_2 > \tilde{p}_1$ when all dealer inventories are roughly the same. In this case, the bid price in the first stage is lower than in the second stage of trading. Similarly, the ask price is higher during customer-dealer trading. Thus, the bid-ask spread is narrower during inter-dealer trading. ∎

2.4 Privately Known Dealer Inventories

In this case, any other dealer's higher private valuation will manifest itself through a lower value of $\tilde{Q}$ in dealer k's private value [notice the appearance of $\tilde{Q}$ in Eq. (9)]. Thus, when the dealers do not know other dealers' inventory positions, the dealers' private values are "affiliated" in the sense of Milgrom and Weber (1982).

Several comments are in order at this point. First, with private but affiliated values, the celebrated *revenue equivalence* among various auction formats no longer holds. In particular, with risk neutral bidders, the seller's expected revenue is higher in an open outcry English auction than in a sealed-bid second-price auction, which is in turn revenue-superior to the sealed-bid first-price auction in this context. Second, with risk averse bidders, these auction formats cannot in general be ranked according to the seller's expected revenue. However, if the bidders have constant absolute risk aversion, as is the case in our model, then the expected revenue in the English auction is at least as large as in the second-price auction.

For concreteness, we will model the customer-dealer trading as a second-price auction.[27]

Proposition 2.3 Suppose all dealer inventories are i.i.d. with a pdf $f(\ )$ and a cdf $F(\ )$ over some interval $[\underline{I}, \bar{I}]$. If the customer-dealer trading is a second-price auction, then dealer k's equilibrium bidding strategy (total payment for a customer order of size $\tilde{z}$) is given by:

$$B_k = \tilde{z}[\bar{v} - \frac{\rho\tau_v^{-1}}{(N-1)^2}(N - \frac{3}{2})(\tilde{z} + 2I_k)] - \frac{N-2}{\rho}[\ln \int_{I_k}^{\bar{I}} e^{\tilde{\kappa}\psi} f(\psi) d\psi],$$

and the customer's expected revenue is:

$$\begin{aligned}\tilde{R}_1 &= \tilde{z}\{\bar{v} - \frac{\rho\tau_v^{-1}}{(N-1)^2}(N - \frac{3}{2})[\tilde{z} + 2N(N-1)\int_{\underline{I}}^{\bar{I}} \eta f(\eta) F(\eta)[1 - F(\eta)]^{N-2} d\eta]\} \\ &\quad - \frac{N(N-1)(N-2)}{\rho}\int_{\underline{I}}^{\bar{I}} [\ln \int_{\eta}^{\bar{I}} e^{\tilde{\kappa}\psi} f(\psi) d\psi] f(\eta) F(\eta)[1 - F(\eta)]^{N-2} d\eta,\end{aligned}$$

[27] To obtain the best price, the customer will likely telephone several market makers and negotiate with them. In this process, information about his order is at least partially revealed to the market makers. In terms of market transparency, it is then reasonable to consider the process of "give-and-take" between the outsider and the market makers to fall somewhere between an open outcry format in which the whole history of bids from all dealers is known by all market participants at any point of time, and a sealed-bid auction which is a more opaque procedure.

where

$$\tilde{\kappa} \equiv \frac{(N-2)\rho^2\tau_v^{-1}}{(N-1)^2}\tilde{z}.$$

Proof: See Appendix B. ∎

The above explicit solution involves the distribution characteristics of the dealer inventories. In general, the payment function B_k is not quadratic in the customer order size $\tilde{z}$, i.e., the unit (bid) price is nonlinear in $\tilde{z}$.[28] Take the example of exponentially distributed dealer inventories with the parameter $\lambda > 0$. That is, $f(\eta) = \lambda e^{-\lambda\eta}$ and $F(\eta) = 1 - e^{-\lambda\eta}$.[29]

For given $\tilde{z}$, the customer's expected revenue is:

$$\begin{aligned}\tilde{R}_1 &= \tilde{z}\{\bar{v} - \frac{\rho\tau_v^{-1}}{(N-1)^2}(N-\frac{3}{2})[\frac{2(2N-1)}{\lambda N(N-1)} + \tilde{z}]\} \\ &\quad + \frac{(N-2)}{\rho}[(\frac{\lambda-\tilde{\kappa}}{\lambda})\frac{2N-1}{N(N-1)} - \ln(\frac{\lambda}{\lambda-\tilde{\kappa}})],\end{aligned}$$

where we require $\tilde{\kappa} < \lambda$. Using Eq. (2), the corresponding expected customer revenue in the benchmark one-shot model is:

$$\tilde{R}_d = \tilde{z}\tilde{p}_d = \tilde{z}[\bar{v} - \rho\tau_v^{-1}\frac{1}{\lambda} - \frac{(N-1)\rho\tau_v^{-1}}{N-2}\frac{\tilde{z}}{N}].$$

In this example, we find that customer revenue is typically higher with inter-dealer trading than without. It is easy to check that the first line of $\tilde{R}_1$ is strictly greater than $\tilde{R}_d$. The second line of $\tilde{R}_1$ is usually positive, except when $\tilde{\kappa}$ approaches λ.[30]

3 Sequential Auctions: Rationalizing "Hot Potato" Trading

Much of the inter-dealer trading in the foreign exchange markets is done via voice-broking and has the following feature that was emphasized in the introduction: The customer trades with dealer 1 who trades with dealer 2 who trades with dealer 3, and so on. The quick sequence of bilateral inter-dealer trades following a customer trade is often referred to as "hot potato"

[28]With known dealer inventories, the unit price is linear in $\tilde{z}$. See Eq. (11).

[29]For ease of computation, we are not imposing the inventory restrictions listed in Section 2.1. With the exponential distribution, inventories can be very large. Hence some dealers could be sellers instead of buyers at the single price that clear the market.

[30]Note that, assuming $\tilde{\kappa}$ is uniformly distributed over $[0, \lambda]$, the expected value of the second line of $\tilde{R}_1$ is strictly positive.

trading. Given our finding in Section 2 that two-stage procedures (unit auction followed by single-price trading) dominates one-stage, single-price procedures, a logical question to ask is whether customer welfare is improved by having more trading rounds.

We construct a sequential trading model of a dealership market where the customer first sells his quantity $\tilde{z}$ to one of N dealers, who then resells a portion of the customer quantity to another dealer, and so on. Each dealer starts with zero inventory. A dealer who has sold some quantity to another dealer cannot trade again. This process continues until there are a total of $m > 3$ dealers left, at which point the dealer who has bought in the previous round resells a fraction of his quantity to the other $m-1$ dealers using a uniform-price auction. Hence the selling dealer in each round chooses a quantity to trade rather than a supply curve. We will refer to this sequential trading process as an (N, m) trading model.

At the n-dealer stage of the (N, m) trading model, the dealer who purchased the quantity q_{n+1} from the previous round, resells a portion of it, q_n, to the other $n-1$ dealers at the price of $p_n(q_n)$. The selling dealer's expected utility will be denoted U_n and the other dealers' expected utility V_n.

We conjecture that the inter-dealer trading prices takes the following form:

$$p_n(q_n) \;=\; \bar{v} - \rho\tau_v^{-1}\lambda_n q_n. \tag{13}$$

The parameter λ_n is an inverse measure of the market liquidity, i.e., the lower is λ_n, the more liquid is the n-dealer stage of inter-dealer trading.

Analyzing the selling dealer's maximization problem:

$$\max_{q_n} U_n \;=\; \bar{v}(q_{n+1} - q_n) - \frac{\rho\tau_v^{-1}}{2}(q_{n+1} - q_n)^2 + q_n p_n(q_n),$$

provides the following optimal quantities:[31]

$$q_n \;=\; \frac{1}{1 + 2\lambda_n} q_{n+1}, \tag{14}$$

$$U_n \;=\; q_{n+1}(\bar{v} - \frac{\lambda_n}{1 + 2\lambda_n}\rho\tau_v^{-1} q_{n+1}). \tag{15}$$

Notice that the above expected utility for the selling dealer is net of the price that he paid for the quantity q_{n+1} obtained in the previous round of trading. For each dealer to be

[31]Since $\lambda_n > 0$ which can be explicitly solved, the second-order condition is always satisfied.

just indifferent between receiving or not receiving q_{n+1}, it must be that:

$$U_n - q_{n+1}p_{n+1}(q_{n+1}) \;=\; V_n.$$

For dealers other than the seller, his expected utility is a constant at any stage of the game, i.e., $V_n = \text{Constant}, \;\forall n$. To determine this constant, we examine the last inter-dealer trading stage (i.e., when $n = m$). Using Eq. (2), the selling price is:

$$p_m(q_m) \;=\; \bar{v} - \frac{(m-2)\rho\tau_v^{-1}}{(m-3)}\frac{q_m}{m-1} \;\equiv\; \bar{v} - \rho\tau_v^{-1}\lambda_m q_m.$$

From the above formula, $\lambda_m = \frac{m-2}{(m-1)(m-3)}$. The expected utility for a dealer who buys in the last round is:

$$\begin{aligned} V_m &= \bar{v}\frac{q_m}{m-1} - \frac{\rho\tau_v^{-1}}{2}(\frac{q_m}{m-1})^2 - p_m(q_m)\frac{q_m}{m-1} \\ &= \frac{\lambda_m - \frac{1}{2(m-1)}}{m-1}\rho\tau_v^{-1}q_m^2. \end{aligned}$$

Thus, using Eq. (14) recursively, we have:

$$\begin{aligned} V_n &= V_m = \frac{\lambda_m - \frac{1}{2(m-1)}}{m-1}\rho\tau_v^{-1}q_m^2 = \frac{\lambda_m - \frac{1}{2(m-1)}}{m-1}\rho\tau_v^{-1}[\frac{q_{m+1}}{(1+2\lambda_m)}]^2 \\ &= \ldots = \frac{\lambda_m - \frac{1}{2(m-1)}}{m-1}\rho\tau_v^{-1}[\frac{q_{n+1}}{(1+2\lambda_m)(1+2\lambda_{m+1})\ldots(1+2\lambda_n)}]^2. \end{aligned}$$

Finally, the price at the $(n+1)$-th stage can be determined as follows:

$$\begin{aligned} p_{n+1}(q_{n+1}) &= \frac{1}{q_{n+1}}(U_n - V_n) \\ &= \bar{v} - \frac{\rho\tau_v^{-1}\lambda_n}{1+2\lambda_n}q_{n+1} - \frac{\lambda_m - \frac{1}{2(m-1)}}{m-1}\frac{\rho\tau_v^{-1}q_{n+1}}{(1+2\lambda_m)^2(1+2\lambda_{m+1})^2\ldots(1+2\lambda_n)^2} \\ &\equiv \bar{v} - \rho\tau_v^{-1}\lambda_{n+1}q_{n+1}. \end{aligned}$$

Thus, we have the following iteration formula for an (N, m) trading game:

$$\lambda_{n+1} \;=\; \frac{\lambda_n}{1+2\lambda_n} + \frac{\lambda_m - \frac{1}{2(m-1)}}{(m-1)(1+2\lambda_m)^2(1+2\lambda_{m+1})^2\ldots(1+2\lambda_n)^2}, \;\forall n \geq 4. \qquad (16)$$

Similarly, we can write down the iteration formula for λ', the liquidity parameter corresponding to an $(N, m+1)$ trading game as follows:

$$\lambda'_{n+1} \;=\; \frac{\lambda'_n}{1+2\lambda'_n} + \frac{\lambda'_{m+1} - \frac{1}{2m}}{m(1+2\lambda'_{m+1})^2(1+2\lambda'_{m+2})^2\ldots(1+2\lambda'_n)^2}, \;\forall n \geq 4, \qquad (17)$$

where $\lambda'_{m+1} = \frac{m-1}{m(m-2)}$. Using the above recursions we can show the following result.

Proposition 3.1 A risk neutral customer prefers a sequential dealership market in which the inter-dealer trading takes place in more stages rather than fewer stages.

Proof: The customer's expected revenue in an (N, m) trading game is:

$$\mathrm{E}[\tilde{R}_{(N,m)}] \;=\; \mathrm{E}[\tilde{z}p_{N+1}(\tilde{z})] \;=\; \int_0^1 (\bar{v} - \rho\tau_v^{-1}\lambda_{N+1}z)zg(z)dz. \tag{18}$$

Similarly, in an $(N, m+1)$ trading game, the expected revenue is:

$$\mathrm{E}[\tilde{R}_{(N,m+1)}] \;=\; \mathrm{E}[\tilde{z}p_{N+1}(\tilde{z})] \;=\; \int_0^1 (\bar{v} - \rho\tau_v^{-1}\lambda'_{N+1}z)zg(z)dz.$$

Since the (N, m) game has one more stage of inter-dealer trading than the $(N, m+1)$ game, we need to show $\mathrm{E}[\tilde{R}_{(N,m)}] > \mathrm{E}[\tilde{R}_{(N,m+1)}]$, or equivalently, $\lambda_{N+1} < \lambda'_{N+1}$. Comparing Eq. (16) with Eq. (17), it is clear that $\lambda_{n+1} < \lambda'_{n+1}, \forall n$, if and only if the following holds:

$$\frac{\lambda_m - \frac{1}{2(m-1)}}{(m-1)(1+2\lambda_m)^2} < \frac{\lambda'_{m+1} - \frac{1}{2m}}{m}.$$

The proposition is proved by verifying the above relation with the substitutions: $\lambda_m = \frac{m-2}{(m-1)(m-3)}$ and $\lambda'_{m+1} = \frac{m-1}{m(m-2)}$. ∎

Propositions 3.1 offers an explanation of why sequential auctions may be beneficial as a trading institution. It demonstrates that more rounds of inter-dealer trading leads to higher expected revenue for the customer. Consequently, it provides a rationale for the "hot potato" phenomenon in the foreign exchange market. A "back-of-envelope" calculation shows that the model can generate trading volumes that rival the substantial inter-dealer trading in the foreign exchange market. With 12 dealers and 10 rounds of inter-dealer trading, the total inter-dealer trading volume is 4.5856 times the initial customer volume, i.e., inter-dealer trading is 82% of total volume. This seems to be in line with the number reported by Lyons (1995) who states that 85% of trading in the foreign exchange market is attributable to inter-dealer trading.

The next result characterizes the evolution of market liquidity, the dealers' trading volume and the equilibrium price in the sequential auctions.

Corollary 3.1 As inter-dealer trading progresses in a sequential trading game (i.e., as n becomes smaller), the market liquidity decreases and the selling dealer retains a larger proportion of the quantity obtained in the previous round. Furthermore, the equilibrium price increases with fewer trading rounds remaining.

Proof: An inverse measure of market liquidity is λ_n. Also, using Eq. (14), the quantity retained in the n-dealer round is:

$$q_{n+1} - q_n = \frac{2\lambda_n}{1+2\lambda_n} q_{n+1}.$$

Thus, for the first statement we need to show that λ_n is decreasing in n.

We can check that $\lambda_{m+1} < \lambda_m$. Now by assuming $\lambda_n < \lambda_{n-1}$, we have:

$$\begin{aligned}
\lambda_{n+1} &= \frac{\lambda_n}{1+2\lambda_n} + \frac{\lambda_m - \frac{1}{2(m-1)}}{(m-1)(1+2\lambda_m)^2(1+2\lambda_{m+1})^2...(1+2\lambda_n)^2} \\
&< \frac{\lambda_{n-1}}{1+2\lambda_{n-1}} + \frac{\lambda_m - \frac{1}{2(m-1)}}{(m-1)(1+2\lambda_m)^2(1+2\lambda_{m+1})^2...(1+2\lambda_{n-1})^2} \\
&= \lambda_n.
\end{aligned}$$

This completes the proof of the first statement by induction.

As for the last statement, using Eq.s (14) and (16) it is easy to show that $\lambda_n q_n$ is increasing in n. Thus, according to Eq. (13), price increases as trading progresses (i.e., as n decreases). ∎

Corollary 3.1 shows that the inter-dealer market becomes more illiquid as trading progresses. This illiquidity of the market occurs because as trading progresses more traders drop out (they have received their desired quantities) and the number of subsequent rounds of trading falls. The lesser the number of rounds of trading, the more difficult is it to share risk with the remaining participants. Thus the liquidity of the market falls. However, since the quantity that is traded also falls, the price paid increases as trading progresses. The effects of sequential inter-dealer trading on market liquidity, transactions volume, and dealer competition are illustrated in Figure 3 and Figure 4.

The sequential auction that we have considered above ends in a final stage (m=4) with the winning dealer trading with 3 dealers who share the good equally. An alternative would be to push the sequential auction analogy further and let the dealer sell via a unit auction to the three remaining dealers. The winning dealer would then run a unit auction for some part of the remaining order flow with the two last dealers. Hence, under this end-game, one dealer gets no quantity allocation at all. We find that, for a risk neutral customer, a "pure" sequential auction that ends with a unit auction is preferred to a sequential dealership market that ends with a share auction. This is consistent with our result that the more rounds of inter-dealer trading the better.

4 Inter-Dealer Trading with a Limit-Order Book

While voice-broking via sequential trading has traditionally been used in the foreign exchange markets, much volume has migrated to electronic limit-order book trading. In particular, as discussed in the introduction, both the EBS partnership and Reuters Dealing 2002 offer systems which have features of a limit-order book. Here we analyze a two-stage model where the inter-dealer competition occurs within a limit-order book which is akin to a discriminatory pricing auction. In our analysis of the limit-order book, we assume that all dealer inventories are identical, $I_k = Q, k = 1, 2, ..., N$. As such, dealer k's trading profit is:

$$\begin{aligned}\tilde{\pi}_k^W &= \tilde{v}\tilde{x}_k^W + \tilde{p}_2'(Q + \tilde{z} - x_k^W) + \sum_{m \neq k}^{N} \int_{\tilde{p}_2'}^{\bar{p}} \tilde{x}_m^L(\psi)d\psi, \\ \tilde{\pi}_k^L &= \tilde{v}(Q + \tilde{x}_k^L) - \tilde{p}_2'\tilde{x}_k^L - \int_{\tilde{p}_2'}^{\bar{p}} \tilde{x}_k^L(\psi)d\psi,\end{aligned}$$

where $\bar{p}$ is the intercept of the demand schedule with the price axis, and $\tilde{p}_2'$ is the market clearing price during the inter-dealer trading stage. We emphasize that, to run the inter-dealer market as an anonymous limit-order book, the customer order in the first stage cannot be disclosed to the dealers who do not receive the order in the first stage.

An important feature of the equilibrium analysis differentiates the results here from those in Section 2. In contrast to the equilibrium in a single-price setting (Section 2) which is independent of distributional assumptions, in this section we focus on the case where the distribution of customer order sizes has a linear hazard ratio. In particular, the pdf and cdf for $\tilde{z} \in [0, 1]$ are the following:

$$g(z) = \frac{1}{\theta}(1 - z)^{\frac{1}{\theta} - 1},$$

$$G(z) = 1-(1-z)^{\frac{1}{\theta}},$$

where θ is a positive parameter related to the moments of the distribution as follows:

$$\mathrm{E}[\tilde{z}] = \frac{\theta}{1+\theta},$$
$$\mathrm{Var}[\tilde{z}] = \frac{\theta^2}{(1+\theta)(1+3\theta+2\theta^2)}.$$

Note that the case of $\theta = 1$ corresponds to the uniform distribution.[32]

4.1 Benchmark: A One-Shot Limit-Order Book

For comparison purposes, a benchmark is presented below where the customer can trade directly with N competing market makers in a one-shot limit-order book setup. The following is a necessary condition for the equilibrium strategies in a limit-order book market:[33]

$$\sum_{j\neq i}^{N} \frac{\partial x_j(p)}{\partial p} = -\frac{\theta(1-z)}{\bar{v}-p-\rho\tau_v^{-1}[x_i(p)+Q]}.$$

A unique symmetric, linear solution to the above equation is the following:

$$x_k(p) = \gamma_b[\bar{v}-\rho\tau_v^{-1}Q - \frac{\rho\tau_v^{-1}\theta}{N(1+\theta)-1} - p], \quad \forall k = 1,2,...,N.$$

where

$$\gamma_b = \frac{N(1+\theta)-1}{(N-1)\rho\tau_v^{-1}}. \tag{19}$$

The equilibrium price and allocations are:

$$\tilde{p}_b = \bar{v}-\rho\tau_v^{-1}Q - \frac{\rho\tau_v^{-1}\theta}{N(1+\theta)-1} - \frac{(N-1)\rho\tau_v^{-1}}{N(1+\theta)-1}\frac{\tilde{z}}{N}, \tag{20}$$
$$\tilde{x}_k = \frac{\tilde{z}}{N}.$$

The customer's total revenue in this case is:

$$\tilde{R}_b = \tilde{p}_b\tilde{z} + \sum_{k=1}^{N}\int_{\tilde{p}_b}^{\bar{p}} x_k(\psi)d\psi.$$

Note that dealer positions at the end of trading are as follows:

$$I_k + \tilde{x}_k = \frac{\tilde{z}}{N} + Q.$$

[32] The only other distribution that yields a linear solution is the exponential distribution, which is a limiting case of the linear-hazard ratio class studied here. This can be seen by taking the limit of $G(z) = 1-(1-\lambda\theta z)^{\frac{1}{\theta}}$ as θ approaches infinity, which is $G(z) = 1-e^{-\lambda z}$.

[33] See Viswanathan and Wang (1997) for a derivation.

4.2 The Second Round: Inter-Dealer Trading

With identical *ex ante* inventories, each dealer has equal probability of winning the customer order in the first stage of trading. Without loss of generality, we designate the dealer who gets to fill the customer order in the first round of trading as dealer W, and all other dealers are referred to as dealer L's. It turns out that for dealer W there is a dominant strategy in the inter-dealer trading stage. In fact, this trading strategy is robust with respect to the distributional assumptions about inventory or customer order size.

Proposition 4.1 If inter-dealer trading is run as a limit-order book, the following is a dominant strategy for dealer W:

$$x^W(p,z) = \begin{cases} \frac{\bar{v}-p}{\rho\tau_v^{-1}} & \text{if } z \in [\underline{s}, 1], \\ Q + z & \text{if } z \in [0, \underline{s}). \end{cases} \tag{21}$$

That is, dealer W will sell a nonzero quantity in the inter-dealer market if and only if the customer order he fills in the first stage is greater than the following threshold size:

$$\underline{s} = \frac{2\theta}{(N-1)(1+\theta) + 2\theta + \sqrt{(N-1)^2(1+\theta)^2 - 4\theta}}. \tag{22}$$

PROOF: See Appendix C. ∎

The analysis of the optimal strategy for a dealer who did not get to fill the customer order, dealer L, involves solving a dynamic optimization problem.

Proposition 4.2 Assume that the dealer inventories are all equal to Q. If inter-dealer trading is run as a limit-order book, the trading strategy for dealer L is:

$$x^L(p) \;=\; \mu_2' - \gamma_2' p, \;\forall L \neq W, \tag{23}$$

where

$$\gamma_2' \;=\; \frac{(N-1)(1+\theta) - 2 + \sqrt{(N-1)^2(1+\theta)^2 - 4\theta}}{2(N-2)\rho\tau_v^{-1}}, \tag{24}$$

$$\mu_2' \;=\; \gamma_2'[\bar{v} - \rho\tau_v^{-1}Q - \frac{2\rho\tau_v^{-1}\theta}{(N-1)(1+\theta) + 2\theta + \sqrt{(N-1)^2(1+\theta)^2 - 4\theta}}]. \tag{25}$$

Dealer L gets a nonzero quantity allocation if and only if the customer order size is greater than $\underline{s}$.

Proof: See Appendix D. ∎

In the case when $z \in [\underline{s}, 1]$, we can solve for the equilibrium price in the second stage as:

$$\tilde{p}_2' = \bar{v} - \rho\tau_v^{-1}Q - \rho\tau_v^{-1}\frac{\frac{2(N-1)\theta}{(N-1)(1-\theta)+\sqrt{(N-1)^2(1+\theta)^2-4\theta}} + z}{1+(N-1)\rho\tau_v^{-1}\gamma_2'}. \tag{26}$$

Therefore, we can express dealer W's quantity allocation after the second stage trading, $\tilde{x}^W$, and dealer L's acquired quantity from inter-dealer trading, $\tilde{x}^L$, as follows:

$$\tilde{x}^W = \frac{\bar{v} - \tilde{p}_2'}{\rho\tau_v^{-1}} \equiv \tilde{X}, \tag{27}$$

$$\tilde{x}^L = \mu_2' - \gamma_2'\tilde{p}_2' - I^L \equiv \tilde{Y} - I^L, \ \forall L \neq W. \tag{28}$$

Thus, the net positions for the dealers at the conclusion of inter-dealer trading are:

$$\begin{aligned} \tilde{x}^W &= \tilde{X}, \\ I^L + \tilde{x}^L &= \tilde{Y}. \end{aligned}$$

Since

$$\begin{aligned} \tilde{X} - \tilde{Y} &= \frac{4N(N-2)(1+\theta)(1-\tilde{z})}{[(N-1)^2(1+\theta) - 2 + (N-1)\sqrt{(N-1)^2(1+\theta)^2-4\theta}]} \\ &\quad \times \frac{1}{[(N-1)(1-\theta) + \sqrt{(N-1)^2(1+\theta)^2-4\theta}]} \ \geq \ 0, \end{aligned} \tag{29}$$

we conclude that the winner of the customer order in the first round retains a fraction of the customer order that is at least as large as what the other dealers get at the end of inter-dealer trading. Therefore, regardless of the nature of pricing rules, two-stage trading always produces a systematic *quantity bias* toward the dealer who fills the customer order in the first round. This distortion is the very reason why dealer competition is more intense in two-stage mechanisms than in one-shot trading.

For a typical dealer k, his (certainty equivalent) utility from inter-dealer trading is:

$$\tilde{U}_k^W = \bar{v}\tilde{X} - \frac{\rho\tau_v^{-1}}{2}\tilde{X}^2 + (Q + \tilde{z} - \tilde{X})\tilde{p}_2' + \frac{1}{2\gamma_2'}\sum_{j\neq k}^{N}(\tilde{Y} - Q)^2,$$

if he gets to fill the customer order in the first stage. If dealer k did not get the outside order during the first round of trading, however, the corresponding utility will be:

$$\tilde{U}_k^L \;=\; \bar{v}\tilde{Y} - \frac{\rho\tau_v^{-1}}{2}\tilde{Y}^2 - (\tilde{Y} - Q)\tilde{p}_2' - \frac{1}{2\gamma_2'}(\tilde{Y} - Q)^2.$$

Therefore,

$$\tilde{U}_k^W - \tilde{U}_k^L \;=\; (\bar{v} - \tilde{p}_2')(\tilde{X} - \tilde{Y}) - \frac{\rho\tau_v^{-1}}{2}(\tilde{X}^2 - \tilde{Y}^2) + \tilde{p}_2'\tilde{z} + \frac{1}{2\gamma_2'}\sum_{j=1}^{N}(\tilde{Y} - Q)^2. \tag{30}$$

Corollary 4.1 When inter-dealer trading is run as a limit-order book, the difference in dealer k's second-stage (certainty equivalent) utility between winning and losing the customer order is given by:

$$\tilde{U}_k^W - \tilde{U}_k^L = \begin{cases} (\bar{v} - \tilde{p}_2')(\tilde{X} - \tilde{Y}) - \frac{\rho\tau_v^{-1}}{2}(\tilde{X}^2 - \tilde{Y}^2) + \tilde{p}_2'\tilde{z} + \frac{1}{2\gamma_2'}\sum_{j=1}^{N}(\tilde{Y} - Q)^2 & \text{if } z \in [\underline{s}, 1], \\ \{\bar{v}(Q + \tilde{z}) - \frac{\rho\tau_v^{-1}}{2}(Q + \tilde{z})^2\} - \{\bar{v}Q - \frac{\rho\tau_v^{-1}}{2}Q^2\} & \text{if } z \in [0, \underline{s}). \end{cases} \tag{31}$$

4.3 The First Round: Customer-Dealer Trading

For any given customer order $\tilde{z}$, Corollary 4.1 provides that the dealers' reservation price has a common value across all dealers, $\tilde{V} \equiv \tilde{U}_k^W - \tilde{U}_k^L$, $(\forall k = 1, 2, ..., N)$.

Proposition 4.3 Assume that the dealer inventories are all equal to Q. If inter-dealer trading is run as a limit-order book, the customer receives the following revenue in the first round of trading:

$$\begin{aligned} \tilde{R}_1' &= \tilde{V} \\ &= \begin{cases} \frac{\rho\tau_v^{-1}}{2}(\tilde{X} - \tilde{Y})^2 + \tilde{p}_2'\tilde{z} + \frac{1}{2\gamma_2'}\sum_{j=1}^{N}(\tilde{Y} - Q)^2 & \text{if } z \in [\underline{s}, 1], \\ \bar{v}\tilde{z} - \frac{\rho\tau_v^{-1}}{2}(2Q\tilde{z} + \tilde{z}^2) & \text{if } z \in [0, \underline{s}). \end{cases} \end{aligned} \tag{32}$$

where the quantity bias toward the winning dealer, $\tilde{X} - \tilde{Y}$, is given by Eq. (29), and the equilibrium price during inter-dealer trading, $\tilde{p}_2'$, by Eq. (26).

Proof: It is straightforward to show:

$$\bar{v} - \tilde{p}_2' - \frac{\rho\tau_v^{-1}}{2}(\tilde{X} + \tilde{Y}) \;=\; \frac{\rho\tau_v^{-1}}{2}(\tilde{X} - \tilde{Y}) \;\geq\; 0.$$

Eq. (32) follows from rewriting Eq. (30) using the last equation and Eq. (29). ∎

Next we compare $\tilde{R}_1'$ with the revenue that a customer receives from a one-shot benchmark model in Subsection 4.1:

$$\tilde{R}_b = \tilde{p}_b\tilde{z} + \frac{1}{2\gamma_b}\sum_{j=1}^{N}(\frac{\tilde{z}}{N})^2, \tag{33}$$

When $\tilde{z} \in [0, \underline{s})$, we can show that $\tilde{R}_1' > \tilde{R}_b$.[34] This can be viewed as an extreme case of the quantity bias toward the winning dealer: winner gets the customer order and no inter-dealer trading takes place afterwards. In this situation, the customer is strictly better off selling to a single dealer who does not retrade rather than selling to N dealers simultaneously.

For customer orders that are not too small ($\tilde{z} \in [\underline{s}, 1]$), there will be inter-dealer trading following the initial customer-dealer trade. In this case, $\tilde{R}_1'$ has an extra term proportional to $(\tilde{X} - \tilde{Y})^2$, which is a *direct* result of the quantity bias for the winning dealer of the customer order. An *indirect* benefit of the quantity bias is related to the fact that dealer competition is intensified, i.e., the equilibrium price during inter-dealer trading is higher than the corresponding price in one-shot book trading. Straightforward comparison of Eq. (26) with Eq. (20) shows that:

$$\tilde{p}_2' - \tilde{p}_b \geq 0.$$

From the last terms in Eq. (32) and (33), we find that there is a cost to two-stage trading related to the use of a flatter demand curve. That is, the competition effect tends to reduce the amount of price-discrimination surplus given up by the dealers. The preceding discussion is summarized in the next result (see Figure 2 for an illustration).

Corollary 4.2 Assume that dealer inventories are all equal to Q. Comparing the customer's revenue from one-shot trading in a limit-order book, $\tilde{R}_b$, with the revenue from a two-stage trading process with a book, $\tilde{R}_1'$, we find that: (i) for small customer order sizes, $\tilde{z} \in [0, \underline{s})$, it is always true that $\tilde{R}_1' > \tilde{R}_b$. For $\tilde{z} \in [\underline{s}, 1]$, however, there are the following tradeoffs: (ii) the benefits to two-stage

[34] To see this, we first show that $\tilde{R}_1' - \tilde{R}_b = F(z)$ is a concave function of z. We then compute $F(0) = 0$, $F(\underline{s}) > 0$, and $F'(0) > 0$, thus $F(z) > 0$, $\forall z \in [0, \underline{s}]$.

trading come from the quantity bias toward the winning dealer and the size of the benefits is small with large outside orders; (iii) there is a cost to two-stage trading due to a lesser amount of price discrimination surplus being extracted from the dealers.

As in the case of inter-dealer trading as a single-price mechanism (see Section 2.3), the benefits of two-stage trading with a book are also directly related to the quantity bias toward the winning dealer and the more intense competition caused by it. The difference, though, is two-fold. First, the cost of a two-step procedure with book trading manifests itself through a reduced amount of price discrimination surplus that the customer can extract from the competing dealers. Second, the benefits of two-stage trading with a book become less important for larger customer order sizes. This is quite different from the situation with a single-price inter-dealer trading, where the competition effect is in proportion to the customer order flow. This difference has its origin in the different way that "bid-reduction" (i.e., departure from pricing according to marginal valuation) operates across a single-price auction and a discriminatory auction: Bid-reduction decreases with quantity levels in a limit-order book, whereas bid-reduction increases with quantity levels in a single-price clearing mechanism.

5 Informative Customer Trades

In this section, we study the impact of private customer information on inter-dealer trading. Conditional on the customer order size, $\tilde{z}$, the dealer is assumed to face a simple inference problem:

$$\mathrm{E}[\tilde{v}|\tilde{z} = z] \;=\; \bar{v} - \xi\rho\tau_v^{-1}z, \tag{34}$$

where ξ is a strictly positive parameter, i.e., a larger customer sell order implies a greater downward adjustment to the expected value of the asset.[35] Since our focus here is on asymmetric information, we set all dealer inventories to zero, i.e., $I_k = 0,\ \forall k = 1, 2, ..., N$.

[35] For simplicity, it is assumed that other aspects of the asset value distribution, such as variance, do not change with $\tilde{z}$.

We assume no disclosure of information about previous trades, which is consistent with trading rules in foreign exchange and other institutional markets.

The linear updating rule allows us to restate the two benchmark one-shot trading models in a convenient way. When the customer order is informative, the only change to the one-shot dealership model is to replace Eq. (1) with:

$$\gamma_d(\xi) \;=\; \frac{N-2}{(N-1)(N\xi+1)\rho\tau_v^{-1}}. \tag{35}$$

Similarly, in the one-shot limit-order book model (assuming the customer order is uniformly distributed, i.e., $\theta = 1$), we can use:

$$\gamma_b(\xi) \;=\; \frac{2N-1}{(N-1)(N\xi+1)\rho\tau_v^{-1}}, \tag{36}$$

in place of Eq. (19). These results are quite intuitive because, with a worsening adverse selection problem (a larger ξ value), the dealers bid less aggressively by steepening their demand curves.

From the preceding discussion, private customer information tends to make the one-shot trading mechanisms less competitive but it does not cause market breakdowns. In other words, linear strategy equilibria always exist in the one-shot models. With two-stage or multiple-stage markets, however, the existence of a linear strategy equilibrium is not assured and the presence of informed customer trades may lead to market breakdowns.[36]

5.1 Sequential Auctions

We first explore the effect of private customer information on the sequential auction model in Section 3. Since a linear updating rule is assumed for the customer-dealer trading stage, we conjecture that in subsequent inter-dealer trading the asset value has a similar correlation structure with the trading quantity. That is:

$$\mathrm{E}[\tilde{v}|q_n] \;=\; \bar{v} - \rho\tau_v^{-1}\xi_n q_n, \quad m \le n \le N+1, \tag{37}$$

at the n-dealer stage of an (N, m) trading model. Note that, by definition, $\xi_{N+1} \equiv \xi$ and $q_{N+1} \equiv z$.

[36]This "no-trade" result is different from other examples of market breakdowns in the literature (see, e.g., Glosten (1989), Bhattacharya and Spiegel (1991)) in that it occurs with two-stage trading but not with one-shot trading.

Suppose inter-dealer trading prices take the form:

$$p_n(q_n) \;=\; \bar{v} - \rho\tau_v^{-1}\lambda_n q_n,$$

we can analyze the selling dealer's maximization problem:

$$\max_{q_n} \; U_n \;=\; (\bar{v} - \rho\tau_v^{-1}\xi_{n+1}q_{n+1})(q_{n+1} - q_n) - \frac{\rho\tau_v^{-1}}{2}(q_{n+1} - q_n)^2 + q_n p_n(q_n),$$

which provides the following optimal quantities:[37]

$$\begin{aligned} q_n &= \frac{1+\xi_{n+1}}{1+2\lambda_n} q_{n+1} \;\equiv\; \delta_n q_{n+1}, \qquad (38)\\ U_n &= q_{n+1}[\bar{v} - \frac{2\lambda_n(1+2\xi_{n+1}) - \xi_{n+1}^2}{2(1+2\lambda_n)}\rho\tau_v^{-1}q_{n+1}]. \end{aligned}$$

Using Eq. (35), the equilibrium strategy in the last stage of the game (where one dealer sells to $m-1$ other dealers) is:

$$x_k(\xi) \;=\; \gamma_d(\xi)(\bar{v} - p),$$

where

$$\gamma_d(\xi) \;=\; \frac{m-3}{(m-2)[(m-1)\xi_m + 1]\rho\tau_v^{-1}}.$$

Thus, the equilibrium price there is:

$$p_m \;=\; \bar{v} - \rho\tau_v^{-1}\lambda_m q_m,$$

with

$$\lambda_m \;=\; \frac{(m-2)[(m-1)\xi_m + 1]}{(m-1)(m-3)}. \qquad (39)$$

In addition, the expected utility for those bidding dealers is:

$$V_m \;=\; \frac{\lambda_m - \xi_m - \frac{1}{2(m-1)}}{m-1}\rho\tau_v^{-1}q_m^2.$$

Using Eq. (38), the above can be rewritten as:

$$\begin{aligned} V_n \;=\; V_m &= \frac{\lambda_m - \xi_m - \frac{1}{2(m-1)}}{m-1}\rho\tau_v^{-1}(q_{m+1}\delta_m)^2\\ = \;...\; &= \frac{\lambda_m - \xi_m - \frac{1}{2(m-1)}}{m-1}\rho\tau_v^{-1}(q_{n+1}\delta_m\delta_{m+1}...\delta_n)^2. \end{aligned}$$

[37]The second-order condition, $1+2\lambda_n > 0$, is satisfied since $\lambda_n > 0$ in equilibrium.

Note that the expected utility for the nonwinning dealers in each stage must be the same.

Thus, the price at the $(n+1)$-th stage can be determined as follows:

$$\begin{aligned} p_{n+1}(q_{n+1}) &= \frac{1}{q_{n+1}}(U_n - V_n) \\ &= \bar{v} - \frac{2\lambda_n(1+2\xi_{n+1}) - \xi_{n+1}^2}{2(1+2\lambda_n)}\rho\tau_v^{-1}q_{n+1} - \frac{\lambda_m - \xi_m - \frac{1}{2(m-1)}}{m-1}(\delta_m\delta_{m+1}...\delta_n)^2\rho\tau_v^{-1}q_{n+1}. \end{aligned}$$

Thus, we derive the following iteration formula for the liquidity parameter:

$$\lambda_{n+1} = \frac{2\lambda_n(1+2\xi_{n+1}) - \xi_{n+1}^2}{2(1+2\lambda_n)} + \frac{\lambda_m - \xi_m - \frac{1}{2(m-1)}}{m-1}(\delta_m\delta_{m+1}...\delta_n)^2. \tag{40}$$

Because the optimal quantity q_n is linear in q_{n+1} (the quantity traded in the previous round, see Eq. (38)), q_n must be linear in z, the original customer order. Hence, the information updating process is a linear one:

$$\begin{aligned} \rho\tau_v^{-1}\xi_n &= -\frac{\mathrm{Cov}(v, q_n)}{\mathrm{Var}(q_n)} = -\frac{\mathrm{Cov}(v, \delta_n\delta_{n+1}...\delta_N z)}{\mathrm{Var}(\delta_n\delta_{n+1}...\delta_N z)} \\ &= -\frac{1}{\delta_n\delta_{n+1}...\delta_N}\frac{\mathrm{Cov}(v,z)}{\mathrm{Var}(z)} = \frac{\rho\tau_v^{-1}\xi_{N+1}}{\delta_n\delta_{n+1}...\delta_N}. \end{aligned}$$

Therefore, we have:

$$\xi_n = \frac{\xi_{N+1}}{\delta_n\delta_{n+1}...\delta_N}. \tag{41}$$

Using the last relation, we have:

$$\delta_n = \frac{\xi_{n+1}}{\xi_n}. \tag{42}$$

From Eq. (38), δ_n is defined as:

$$\delta_n = \frac{1+\xi_{n+1}}{1+2\lambda_n}. \tag{43}$$

Thus, ξ_{n+1} can be solved from the last two relations as:

$$\xi_{n+1} = \frac{\xi_n}{1+2\lambda_n - \xi_n}, \tag{44}$$

which provides the iteration formula for the information parameter of the model. Note that, using Eq. (42), we can also rewrite Eq. (40) as:

$$\lambda_{n+1} = \frac{2\lambda_n(1+2\xi_{n+1}) - \xi_{n+1}^2}{2(1+2\lambda_n)} + \frac{\lambda_m - \xi_m - \frac{1}{2(m-1)}}{m-1}(\frac{\xi_{n+1}}{\xi_m})^2. \tag{45}$$

Corollary 5.1 As inter-dealer trading progresses in a sequential trading game (i.e., as n becomes smaller), both the adverse selection problem and market liquidity worsen (i.e., $\xi_{n+1} < \xi_n$ and $\lambda_{n+1} < \lambda_n$).

PROOF: See Appendix E. ∎

Proposition 5.1 When the informativeness of the order flow is sufficiently high, market breakdowns will always occur in sequential auctions. Fixing the total number of dealers, the parameter region with market breakdowns expands with the number of trading rounds.

PROOF: Substituting Eq. (39) into Eq. (44), we have:

$$\xi_{m+1} = \frac{1}{\frac{m-1}{m-3} + \frac{m^2-2m-1}{(m-1)(m-3)\xi_m}} < \frac{m-3}{m-1}.$$

as long as ξ_m is positive and $m \geq 4$. Using Corollary 5.1, we then get $\xi = \xi_{N+1} < \xi_N < ... < \xi_{m+1} < (m-3)/(m-1)$.

Thus, for any given (N, m) sequential auction model, we can find an initial information parameter that violates the above relations (e.g., by choosing $\xi \geq (m-3)/(m-1)$). This shows that market breakdowns will occur when there is a severe adverse selection problem at the customer-dealer trading stage.

As for the second statement, we observe that, with more trading rounds (smaller m values), there are more ξ values that satisfy the market breakdown relation $\xi > (m-3)/(m-1)$. ∎

Hence with informative order flow, the sequential auction mechanism suffers from market breakdown problems. The intuition for market breakdowns is as follows. Because only the winning dealer observes the customer order flow in the first round, he attempts to manipulate the beliefs of the other agents in the second round. This tends to steepen the bidding curves submitted by the other agents. The extent of information asymmetry is worsened along the auction path. At high enough values of the initial adverse selection parameter ξ, the inference parameter in the last round (ξ_m) becomes negative, indicating the nonexistence of a linear strategy equilibrium.

Not surprisingly, the region of breakdowns becomes larger when there are more trading rounds and is the least when there is one round of inter-dealer trading. In each round of inter-dealer trading, the winning dealer from the prior round attempts to manipulate the beliefs of the other dealers. The more rounds of trading we have, the greater is the impact of this belief manipulation on the market liquidity. Hence with enough initial information asymmetry and enough rounds of trading, we find that the final round of trading collapses. With more initial information asymmetry, the two final rounds of trading collapse. This is shown above in Proposition 5.1 and illustrated in Figure 6.

Contrasting the above result with Proposition 3.1, we see that tension exists between the bargaining advantage implied by running more auctions and the information disadvantage associated with more inter-dealer trading. This implies that some interior number of rounds can be optimal. Figure 6 illustrates this.

5.2 Two-Stage Limit-Order Book Trading

Given the increasing importance of limit-order book in the context of inter-dealer trading, it is important to understand how limit-order book trading is affected by the presence of private information. For this purpose, we modify the model in Section 4 by adding a linear inference problem (Eq. (34)) to the customer-dealer trading stage.

Proposition 5.2 Assume that customer orders are uniformly distributed over the interval $[0, 1]$.[38] If a limit-order book is used during inter-dealer trading, there exists a linear strategy equilibrium in which the strategy for the winning dealer of the customer order is:

$$x^W(p,z) = \begin{cases} \frac{\bar{v} - \xi\rho\tau_v^{-1}z - p}{\rho\tau_v^{-1}} & \text{if } z \in [\underline{s}, 1], \\ z & \text{if } z \in [0, \underline{s}). \end{cases} \tag{46}$$

The inter-dealer trading strategy for dealer L is $x^L(p) = \mu - \gamma p$, where

$$\gamma = \frac{2(N-2) - N\xi + \sqrt{4N(N-2)(1+\xi) + N^2\xi^2}}{2(N-2)\rho\tau_v^{-1}(1+N\xi)},$$

[38]The extension to the class of linear hazard ratio distributions is straightforward but notationally cumbersome.

$$\begin{aligned}
\mu &= \frac{(1+\frac{\sigma}{\rho\tau_v^{-1}})\bar{v} - \sigma(1+\xi)}{\rho\tau_v^{-1}(1+\xi) + (N-1)(\rho\tau_v^{-1}\xi - \sigma)}, \\
\underline{s} &= \frac{\gamma\bar{v} - \mu}{\rho\tau_v^{-1}(1+\xi)\gamma}, \\
\sigma &= \frac{1}{(N-2)\gamma + \frac{1}{\rho\tau_v^{-1}}}.
\end{aligned} \tag{47}$$

PROOF: See Appendix F. ∎

It is interesting to compare the way market makers in different inter-dealer trading systems respond to the problem of private information. In a dealership setting (e.g., a sequential auction), the dealers use increasingly inelastic demand curves when adverse selection worsens. This phenomenon eventually leads to market breakdowns. With a limit-order book, however, the winning dealer in the customer-dealer round makes two adjustments in response to asymmetric information: lowering the intercept of his demand curve, and decreasing the "no-trade" zone (i.e., increasing $\underline{s}$). It turns out that, for customer orders above a certain threshold size, a linear strategy equilibrium always exists in limit-order book trading.

The above result demonstrates that, when inter-dealer trading takes place in a limit-order book, a linear strategy equilibrium exists even when there is a severe adverse selection problem. This stands in contrast with the susceptibility to private information of dealership inter-dealer trading systems. It suggests that, in environments where the concentration of informed traders is expected to be high, inter-dealer trading might well take the form of a limit-order book rather than a dealership structure.

5.3 The Customer's Expected Revenue

To a large extent, the successes and failures of inter-dealer trading systems are measured by the customer welfare achievable under such systems. Thus we are interested in comparing the expected customer revenue under both the auction mechanism and the limit-order book structure when private information may be an issue. The revenue comparisons in this section are established through numerical computation based on the following relations. We set all dealer inventories to be zero.

From Eq. (18), the expected customer revenue in a sequential auction is:

$$\mathrm{E}[\tilde{R}_S] \;=\; \int_0^1 [\bar{v} - \rho\tau_v^{-1}\lambda_{N+1}z]zg(z)dz. \tag{48}$$

Using (32), the customer's expected revenue in limit-order book trading is:

$$\mathrm{E}[\tilde{R}_B] \;=\; \int_{\underline{s}}^1 [\frac{\rho\tau_v^{-1}}{2}(X-Y)^2 + pz + \frac{N}{2\gamma}Y^2]g(z)dz + \int_0^{\underline{s}} (\bar{v} - \frac{\rho\tau_v^{-1}}{2}z)zg(z)dz, \tag{49}$$

where:

$$\begin{aligned} p &= \frac{\bar{v} + (N-1)\rho\tau_v^{-1}\mu - (1+\xi)\rho\tau_v^{-1}z}{1+(N-1)\rho\tau_v^{-1}\gamma}, \\ X &= \frac{\bar{v} - \xi\rho\tau_v^{-1}z - p}{\rho\tau_v^{-1}}, \\ Y &= \mu - \gamma p. \end{aligned}$$

The following result summarizes the impact of inter-dealer trading system designs on customer welfare in the absence of asymmetric information. See Figure 5 for illustration.

Proposition 5.3 Suppose the customer trades do not carry private information and there is no public disclosure of trades. In a two-stage trading environment, a risk neutral customer prefers the inter-dealer trading to occur in a limit-order book setting rather than in a dealership setting. However, a dealership market functioning as a sequence of auctions yields the customer a higher expected revenue than a two-stage limit-order book market when the number of auctions is sufficiently large.

The intuition for Proposition 5.3 derives from results in Sections 2 and 3 where we demonstrate that, in the absence of private information, having more rounds of inter-dealer trading tends to enhance the customer's welfare. Because of the possibility of reselling portions of the customer order in subsequent inter-dealer trading, the dealers engage in more intense competition during the initial stage of customer-dealer trading. This of course works in favor of the original seller – the customer. The reason three rounds of trading is usually better than two rounds of trading is as follows. With two more rounds of inter-dealer trading to go, the dealers competing for the customer order in the first stage realize that, upon winning, they do not have to trade as aggressively as they would if there is only one more round to go.[39] Since all dealers are assumed to be symmetric on an *ex ante* basis, the benefits associated with more rounds of trading accrue to the customer and not the competing dealers.

Proposition 5.4 The customer's expected revenues in both the auction market and the limit-order book market decrease when the customer trades become more informative of asset value. When the information asymmetry is significant (i.e., at larger values of ξ), a risk neutral customer favors the two-stage limit-order book mechanism over the sequential auction market.

From Figure 7, it is clear that both $\mathrm{E}[\tilde{R}_S]$ and $\mathrm{E}[\tilde{R}_B]$ are decreasing functions of ξ. That is, the presence of asymmetric information has a negative impact on the customer welfare in the dealership market as well as in limit-order book trading. However, the speed at which private information adversely affects the customer's expected revenue is different depending on the structure of inter-dealer trading.

When private information is nonexistent or unimportant (zero or small ξ values), the auction market (two or more stages) tends to be favored by a risk neutral customer to the two-stage limit-order book mechanism. When private information is pervasive (large values of ξ), however, the limit-order book is far superior to the dealership structure irregardless of the number of auction rounds the inter-dealer trading may take. Proposition 5.4 reinforces the notion that limit-order book is a better venue for inter-dealer trading when there is a severe adverse selection problem.

6 Conclusion

In this paper, we study whether multi-stage trading mechanisms that involve inter-dealer trading provide welfare improvement for the customers over the one-shot settings traditionally analyzed in the market microstructure literature. Important determinants of such a comparison include the pricing rules in the inter-dealer market, the size and distribution

[39] Corollary 3.1 confirms that, as the auction progresses, the selling dealers become more aggressive in retaining a larger fraction of their previously acquired quantities.

of the customer orders, the information content of the customer orders, and in the case of sequential auction market, the number of rounds of trading.

We identified two main advantages of multi-stage processes that are absent in one-shot trading environments. The first advantage has to do with the fact that winning dealers of the customer order retain an above average share. This provides strong incentives for all dealers to engage in more heightened competition for the customer order initially. This intensified competition leads to improved customer welfare relative to a one-shot mechanism. A second advantage of multi-stage procedures relates to their ability to better deal with "collusive" bidding among market makers. The combination of an "all-or-nothing" first stage bidding and an active inter-dealer trading phase serves the dual purpose of curbing strategic bidding and attaining satisfactory risk sharing among the dealers.

A sequential auction model is analyzed which approximates the traditional voice-broking in foreign exchange trading. In the sequential auction, there is repeated bilateral trading between dealers which is consistent with the "hot potato" view of foreign exchange trading. A back-of-the-envelope calculation shows that the model generates inter-dealer volumes consistent with the 85% of total trading volume reported in the literature. We show that such a procedure generally dominates two-stage dealership market and two-stage limit-order book market. In fact, in the absence of private information, customer revenue increases with more trading rounds.

When the order flow is informative, however, sequential trading also leads to the winning dealer attempting to manipulate the perceptions of the losing dealers with regard to the underlying asset value which is assumed to correlate with the actual customer order size. Although in equilibrium all information is revealed, this distorts risk sharing and works against the multi-stage mechanism. As a result, the customer's expected revenue in multi-stage markets goes down with increasing information content in customer orders. Of the two forms of inter-dealer trading, the single-price dealership structure is found to be more susceptible to adverse selection problems than the limit-order book. In fact, market breakdown always occurs in sequential dealership trading when asymmetric information is high. We interpret this as an argument for the use of limit-order book for inter-dealer trading.

Our results thus provide strong support for inter-dealer mechanisms that utilize a sequential auction procedure or a limit-order book. The first of these procedures corresponds to traditional voice-broking services in the foreign exchange market while the second is closer to the electronic limit-order book system that is run by Reuters or the EBS partnership. We believe that more theoretical and empirical work in the foreign exchange market and more attention to the empirical data are needed given the importance of this market.

A Proof of Proposition 2.1

Given the information partition of the dealers, we conjecture the following linear equilibrium strategies:

$$\begin{aligned} x^W &= \mu' - \gamma' p + \beta'(I^W + z), \\ x^L &= \mu - \gamma p - \beta I^L, \ \forall L \neq W. \end{aligned}$$

Note that dealer W can condition his strategy on z, but dealer L cannot because the actual customer order size is known to the winning dealer in the first round of trading only.

From the market clearing condition

$$I^W + z = x^W + \sum_{L \neq W}^{N} x^L,$$

we can back out W's residual supply curve:

$$p = \lambda[x^W + (N-1)\mu - I^W - z - \beta \sum_{L \neq W}^{N} I^L],$$

where

$$\lambda = \frac{1}{(N-1)\gamma}. \tag{50}$$

Since dealer W already possesses $I^W + z$ units of asset to begin with, he will receive a payment on the $I^W + z - x^W$ units that he resells to the other dealers during inter-dealer trading. For any given set of $I^L, \ L \neq W$, dealer W's trading profit is therefore:

$$\tilde{\pi}^W = \tilde{v} x^W + p(I^W + z - x^W). \tag{51}$$

Expected utility maximization leads to the following first-order condition for W:[40]

$$0 = \bar{v} - p + \lambda(I^W + z - x^W) - \rho \tau_v^{-1} x^W, \tag{52}$$

thus:

$$x^W = \frac{\bar{v} - p + \lambda(I^W + z)}{\rho \tau_v^{-1} + \lambda}. \tag{53}$$

[40]Since the objective function is concave in x^W, the first-order condition is necessary and sufficient.

Comparing the above equation with the conjectured functional form, we have:

$$\mu' = \frac{\bar{v}}{\rho\tau_v^{-1} + \lambda}, \tag{54}$$

$$\beta' = \frac{\lambda}{\rho\tau_v^{-1} + \lambda}, \tag{55}$$

$$\gamma' = \frac{1}{\rho\tau_v^{-1} + \lambda}. \tag{56}$$

Note that the solution obtained does not explicitly depend on I^L, thus, it is optimal for all realizations of $I^L,\ L \neq W$.

The analysis of dealer L's optimal strategy proceeds along similar lines, the only significant difference being that dealer L cannot condition his strategy on z. In the proof below, however, we choose a specific realization of $\tilde{z}$ and then find the corresponding trading strategy for dealer L. It turns out that this strategy is independent of z and thus constitutes an optimal strategy for all possible values of $\tilde{z}$.

From dealer L's perspective, the market clearing condition:

$$I^W + z = x^L + x^W + \sum_{k \neq L,W}^{N} x_k,$$

can be used to back out his residual supply curve:

$$p = \sigma[x^L + (N-2)\mu - I^W - z - \beta \sum_{k \neq L,W}^{N} I_k + \mu' + \beta'(I^W + z)],$$

where

$$\sigma = \frac{1}{\gamma' + (N-2)\gamma}. \tag{57}$$

If he acquires x^L units in inter-dealer trading to augment his initial inventory, dealer L will have the following trading profit:

$$\tilde{\pi}^L = \tilde{v}(I^L + x^L) - px^L. \tag{58}$$

Expected utility maximization leads to the following first-order condition for L:

$$0 = \bar{v} - p - \sigma x^L - \rho\tau_v^{-1}(I^L + x^L),$$

thus:

$$x^L = \frac{\bar{v} - p - \rho\tau_v^{-1} I^L}{\rho\tau_v^{-1} + \sigma}. \tag{59}$$

Comparing the above equation with the conjectured functional form, we have:

$$\mu = \frac{\bar{v}}{\rho\tau_v^{-1} + \sigma}, \tag{60}$$

$$\beta = \frac{\rho\tau_v^{-1}}{\rho\tau_v^{-1} + \sigma}, \tag{61}$$

$$\gamma = \frac{1}{\rho\tau_v^{-1} + \sigma}. \tag{62}$$

The coefficients (μ', β', γ') and (μ, β, γ) can be solved from Eq.s (50), (54), (55), (56), (57), (60), (61), and (62). The solutions are summarized in the proposition. ∎

B Proof of Proposition 2.3

First we prove the following lemma which establishes that it is appropriate to use the difference in certainty equivalent utilities from the second stage (depending on whether the dealer wins or loses in the first round) as the private values in modeling the first round trading as a unit-demand auction.

Lemma B.1 Optimal trading strategies in the first round can be determined by assuming that each market maker has the private value, $\tilde{V}_k$, which is equal to the difference in certainty equivalent utility levels when he wins the customer order and when he does not.

Proof: Because a dealer uses demand curve as his trading strategy, the dealer can condition the total payment he makes to the customer (upon winning) on the size of the customer order. Consequently, in the following we will integrate over the true value of the asset, $\tilde{v}$, and other dealers' inventories, $\tilde{I}_{-k}$, but not over $\tilde{z}$.

Let δ be an indicator function which takes on value 1 if dealer k wins the customer order, and value 0 otherwise. Denoting the equilibrium bidding strategy in a second-price auction as B_k and the equilibrium price in the second-round as $\tilde{p}_2$, then dealer k's objective function in the first round is:

$$\begin{aligned}
& \mathrm{E}_{\tilde{v},\tilde{I}_{-k}}[-e^{-\rho\{\delta[\tilde{v}\tilde{x}_k^W+\tilde{p}_2(I_k+\tilde{z}-\tilde{x}_k^W)-B_k]+(1-\delta)[\tilde{v}(I_k+\tilde{x}_k^L)-\tilde{p}_2\tilde{x}_k^L]\}}] \\
= \; & \mathrm{E}_{\tilde{v},\tilde{I}_{-k}}[-e^{-\rho\tilde{v}[\delta\tilde{x}_k^W+(1-\delta)(I_k+\tilde{x}_k^L)]-\rho\delta[\tilde{p}_2(I_k+\tilde{z}-\tilde{x}_k^W)-B_k]-\rho(1-\delta)\tilde{p}_2\tilde{x}_k^L}] \\
= \; & \mathrm{E}_{\tilde{I}_{-k}}[-e^{-\rho\{\bar{v}[\delta\tilde{x}_k^W+(1-\delta)(I_k+\tilde{x}_k^L)]-\frac{\rho\tau_v^{-1}}{2}[\delta\tilde{x}_k^W+(1-\delta)(I_k+\tilde{x}_k^L)]^2\}-\rho\delta[\tilde{p}_2(I_k+\tilde{z}-\tilde{x}_k^W)-B_k]-\rho(1-\delta)\tilde{p}_2\tilde{x}_k^L}] \\
= \; & \mathrm{E}_{\tilde{I}_{-k}}[-e^{-\rho\delta(\tilde{U}_k^W-B_k)-\rho(1-\delta)\tilde{U}_k^L}] \\
= \; & \mathrm{E}_{\tilde{I}_{-k}}[-e^{-\rho\tilde{U}_k^L}]\mathrm{E}_{\tilde{I}_{-k}}[e^{-\rho\delta(\tilde{U}_k^W-\tilde{U}_k^L-B_k)}].
\end{aligned}$$

Notice that $\tilde{U}_k^L$ is independent of $\tilde{I}_{-k}$ and thus we are left with an objective function which is proportional to the objective function of a bidder in a standard single-unit auction with private value of $\tilde{U}_k^W - \tilde{U}_k^L$.

The above proof assumes that the inter-dealer trading occurs at a single price. It can be easily adapted to show that the certainty equivalent approach is also valid when inter-dealer trading is a limit-order book or when the customer order is informative of the value of the asset according to Eq. (34). ∎

Suppose dealer l holds the smallest inventory among all dealers other than k. Let m be any dealer who is neither k nor l. A result from Milgrom and Weber (1982) (page 1114) states that the equilibrium bidding strategy in a second-price auction as a function of private values, $B_k(x)$, is the unique solution to:

$$\mathrm{E}[-e^{-\rho[\tilde{V}_k-B_k(x)]}|I_k=x, I_l=x, I_m \leq x, \forall m \neq k,l] \;=\; -e^{-\rho\times 0}, \tag{63}$$

where $\tilde{V}_k$ is bidder k's private value of the asset, given by Eq. (10):

$$\tilde{V}_k \;=\; \tilde{z}\{\bar{v} - \frac{\rho\tau_v^{-1}}{(N-1)^2}[I_k + N(N-2)\tilde{Q} + (N-\frac{3}{2})\tilde{z}]\}.$$

For any given $\tilde{z}$, Eq. (63) can be inverted to give:

$$B_k(x) \;=\; -\frac{1}{\rho}\ln \mathrm{E}[e^{-\rho\tilde{V}_k}|I_k=x, I_l=x, I_m \geq x]$$

$$
\begin{aligned}
&= -\frac{1}{\rho}\ln \mathrm{E}[e^{-\rho\tilde{z}\{\bar{v}-\frac{\rho\tau_v^{-1}}{(N-1)^2}[\tilde{I}_k+(N-2)(\tilde{I}_k+\tilde{I}_l+\sum_{m\neq k,l}^{N}\tilde{I}_m)+(N-\frac{3}{2})\tilde{z}]\}}|I_k=x, I_l=x, I_m\geq x] \\
&= \tilde{z}\{\bar{v}-\frac{\rho\tau_v^{-1}}{(N-1)^2}[x+2(N-2)x+(N-\frac{3}{2})\tilde{z}]\} \\
&\quad -\frac{N-2}{\rho}[\ln\int_x^{\bar{I}} e^{\frac{(N-2)\rho^2\tau_v^{-1}}{(N-1)^2}\tilde{z}\psi} f(\psi)d\psi].
\end{aligned}
$$

Therefore, dealer k's bidding strategy (i.e., total payment for a customer order of size $\tilde{z}$) expressed as a function of his inventory is:

$$
B_k(I_k) = \tilde{z}[\bar{v}-\frac{\rho\tau_v^{-1}}{(N-1)^2}(N-\frac{3}{2})(2I_k+\tilde{z})] - \frac{N-2}{\rho}[\ln\int_{I_k}^{\bar{I}} e^{\frac{(N-2)\rho^2\tau_v^{-1}}{(N-1)^2}\tilde{z}\psi} f(\psi)d\psi].
$$

The seller's expected revenue is taking the expectation of the above expression over $\tilde{I}_k=\tilde{I}_2$ (the second-lowest inventory):

$$
\begin{aligned}
\tilde{R}_1 &= \tilde{z}[\bar{v}-\frac{\rho\tau_v^{-1}}{(N-1)^2}(N-\frac{3}{2})(2\bar{I}_2+\tilde{z})] \\
&\quad -\frac{N(N-1)(N-2)}{\rho}\int_{\underline{I}}^{\bar{I}}[\ln\int_{\eta}^{\bar{I}} e^{\frac{(N-2)\rho^2\tau_v^{-1}}{(N-1)^2}\tilde{z}\psi} f(\psi)d\psi]f(\eta)F(\eta)[1-F(\eta)]^{N-2}d\eta,
\end{aligned}
$$

where $\bar{I}_2$ denotes the expected second-lowest inventory level $N(N-1)\int_{\underline{I}}^{\bar{I}}\eta f(\eta)F(\eta)[1-F(\eta)]^{N-2}d\eta$. ∎

C Proof of Proposition 4.1

Suppose the dealer Ls play the strategy in Proposition 4.2, $x^L(p)$, which does not depend on the realization of $\tilde{z}$ (only dealer W observes the true customer order size going into the inter-dealer trading stage).

If dealer W intends to retain $x^W < Q+z$ units after the second round of trading, then his inter-dealer trading profit is:

$$
\tilde{\pi}^W = \tilde{v}x^W + p(Q+z-x^W) + \sum_{j\neq W}^{N}\int_p^{\bar{p}} x_j(\psi)d\psi, \tag{64}
$$

where $\bar{p}$ is the intercept of dealer j's demand schedule with the price axis.

Maximization of dealer W's expected utility is equivalent to maximizing the following objective function:

$$\bar{v}x^W + p(Q + z - x^W) - \frac{\rho\tau_v^{-1}}{2}(x^W)^2 + \sum_{j\neq W}^{N}\int_p^{\bar{p}} x_j(\psi)d\psi,$$

which leads to the first-order condition for W:

$$\begin{aligned} 0 &= \bar{v} - p + \frac{\partial p}{\partial x^W}(Q + z - x^W) - \rho\tau_v^{-1}x^W - \frac{\partial p}{\partial x^W}\sum_{j\neq W}^{N} x_j(p) \\ &= \bar{v} - p - \rho\tau_v^{-1}x^W. \end{aligned}$$

The market clearing condition $Q + z = x^W + \sum_{j\neq W}^{N} x_j(p)$ is used in the last step.

Thus far, we have proved that the trading strategy:

$$x^W(p) = \frac{\bar{v} - p}{\rho\tau_v^{-1}},$$

is a dominant strategy (i.e., optimal with respect to any trading strategy on the part of the dealer Ls) so long as the dealer Ls can obtain positive quantity allocation from inter-dealer trading.

In the next appendix, we will show that there exists a unique cutoff customer order size, $\underline{s}$ such that dealer W is a net seller during inter-dealer trading if and only if $z \in [\underline{s}, 1]$. Should dealer W decide to use the above strategy $x^W(p)$ when he observes a z value that is less than $\underline{s}$ (based on the incorrect assumption that all other dealers get a positive amount), it can be shown that dealer Ls' quantity allocation would be negative in that case. This cannot happen in equilibrium and, therefore, dealer W retains all of the customer order he fills if it is sufficiently small (i.e., when $z < \underline{s}$). ∎

D Proof of Proposition 4.2

The proof in this appendix consists of three parts. By conjecturing that dealer W will trade in the inter-dealer market only if the customer order is greater than a threshold size $0 < \underline{s} < 1$, we first establish an ordinary differential equation (ODE) as the necessary condition for dealer L's equilibrium trading strategies. Then we explicitly solve for a linear solution to the ODE. In the last step, we verify the existence of such a threshold customer order size, $\underline{s}$.

In the following we characterize dealer i's optimal trading strategy, $x_i(p)$ (we sometimes write dealer i's upward-sloping residual supply curve as $h(p)$) in response to the strategy used by the other dealers, $x_j(p)$ ($\forall j \neq i$). We use $g(z)$ and $G(z)$ to denote the pdf and cdf for the customer order size $\tilde{z}$ when it falls within $[\underline{s}, 1]$.

We can write dealer i's uncertain trading profit as follows:

$$\tilde{\pi}_i \quad = \quad \tilde{v} x_i(p) - TP_i, \quad i \neq W,$$

where his total payment is:

$$\begin{aligned} TP_i &= (1-\alpha) p x_i(p) + \int_0^{x_i(p)} p(q) dq \\ &= (1-\alpha) p x_i(p) + [\sum_{j=1}^{N} \int_0^{x_j(p)} p(q) dq - \sum_{j \neq i}^{N} \int_0^{x_j(p)} p(q) dq] \\ &\equiv (1-\alpha) p x_i(p) + [A - B]. \end{aligned}$$

We note the following relations for use in subsequent calculation:

$$\frac{\partial A}{\partial z} \quad = \quad \sum_{j=1}^{N} p \frac{\partial x_j}{\partial z} \quad = \quad p \frac{\partial (Q+z)}{\partial z} \quad = \quad p,$$

and

$$\frac{\partial B}{\partial p} \quad = \quad \sum_{j \neq i}^{N} p \frac{\partial x_j(p)}{\partial p} \quad = \quad -p \frac{\partial h(p)}{\partial p}, \tag{65}$$

where we made use of the market clearing condition:

$$Q + z = x^W + \sum_{i \neq W}^{N} x_i.$$

We assume that dealer i chooses his optimal trading strategy by maximizing the following derived mean-variance utility function:

$$\mathrm{E}_{\tilde{z}}[\bar{v} h(p) - \frac{\rho \tau_v^{-1}}{2} h^2(p) - TP_i],$$

Defining $A(z)$ as the state variable and $p(z)$ the control variable, we can analyze the problem using the following Lagrangian:

$$L = g(z)[\bar{v}h(p) - \frac{\rho\tau_v^{-1}}{2}h^2(p) - A(z) + B(p)] + \lambda p.$$

The optimality condition is:

$$0 = \frac{\partial L}{\partial p} = g(z)[\bar{v} - p - \rho\tau_v^{-1}h(p)]\frac{\partial h(p)}{\partial p} + \lambda. \quad (66)$$

The adjoint equation is:

$$-\frac{\partial \lambda}{\partial z} = \frac{\partial L}{\partial A} = -g(z), \quad (67)$$

with the transversality condition being

$$\lambda(\underline{s}) = -1.$$

Using the transversality condition, the adjoint equation can be integrated to obtain

$$\lambda(z) = -[1 - G(z)]. \quad (68)$$

Combining Eq.s (66) and (68), we have:

$$[\bar{v} - p - \rho\tau_v^{-1}h(p)]\frac{\partial h(p)}{\partial p} = \frac{1 - G(z)}{g(z)}, \quad (69)$$

which is equivalent to:

$$\sum_{j\neq i}^{N} x_j'(p) = -\frac{[1 - G(z)]/g(z)}{\bar{v} - p - \rho\tau_v^{-1}x_i(p)}. \quad (70)$$

Motivated by dealer W's use of a linear bidding strategy, we now search for a linear strategy equilibrium of the form:

$$x_k = \mu - \gamma p, \quad (71)$$

for dealers k. With this, the market clearing condition is:

$$Q + \tilde{z} = x^W + x_k + (N - 2)(\mu - \gamma p).$$

The hazard ratio for $\tilde{z}$ over the interval $[\underline{s}, 1]$ is:

$$\frac{1-G(z)}{g(z)} \;=\; \theta(1-z) \;=\; \theta[Q+1-x^W-x_k-(N-2)(\mu-\gamma p)]. \qquad (72)$$

Plugging Eq. (21), (71), and (72) into Eq. (70), we obtain:

$$\frac{1}{\rho\tau_v^{-1}}+(N-2)\gamma \;=\; \frac{\theta[Q+1-\frac{\bar{v}-p}{\rho\tau_v^{-1}}-(N-1)(\mu-\gamma p)]}{\bar{v}-p-\rho\tau_v^{-1}(Q+x_k)},$$

which can also be written as:

$$x_k \;=\; \frac{1}{\rho\tau_v^{-1}}[(1+\frac{\theta\sigma}{\rho\tau_v^{-1}})(\bar{v}-p)+\theta\sigma(N-1)(\mu-\gamma p)-\theta\sigma(Q+1)-\rho\tau_v^{-1}Q],$$

with

$$\sigma = \frac{1}{\frac{1}{\rho\tau_v^{-1}}+(N-2)\gamma}.$$

Because of the symmetry among the $N-1$ dealers other than dealer W, we must have:

$$\begin{aligned}\gamma \;&=\; \frac{1}{\rho\tau_v^{-1}}[(1+\frac{\theta\sigma}{\rho\tau_v^{-1}})+\theta\sigma(N-1)\gamma],\\ \mu \;&=\; \frac{1}{\rho\tau_v^{-1}}[(1+\frac{\theta\sigma}{\rho\tau_v^{-1}})\bar{v}+\theta\sigma(N-1)\mu-\theta\sigma(Q+1)-\rho\tau_v^{-1}Q].\end{aligned}$$

The solutions to the above equations are:

$$\begin{aligned}\gamma \;&=\; \frac{(N-1)(1+\theta)-2+\sqrt{(N-1)^2(1+\theta)^2-4\theta}}{2(N-2)\rho\tau_v^{-1}},\\ \mu \;&=\; \gamma[\bar{v}-\rho\tau_v^{-1}Q-\frac{2\rho\tau_v^{-1}\theta}{(N-1)(1+\theta)+2\theta+\sqrt{(N-1)^2(1+\theta)^2-4\theta}}].\end{aligned}$$

Given the above solutions, it can be checked that the following choice of threshold customer order size:

$$\underline{s} = \frac{2\theta}{(N-1)(1+\theta)+2\theta+\sqrt{(N-1)^2(1+\theta)^2-4\theta}} \in (0,1),$$

is a unique number that satisfies the requirements that $Q+z \geq x^W$ and $x_k \geq 0, \forall k \neq W$ for $\tilde{z} \in [\underline{s}, 1]$.

This last statement completes the proofs of Propositions 3.1 and 3.2. ∎

E Proof of Corollary 5.1

We assume that $\xi_n > 0$, $\forall n$. Using Eq. (44), we note that the result $\xi_{n+1} < \xi_n$ follows directly from $\lambda_n > \xi_n$. Thus, we first prove that $\lambda_n > \xi_n$ by induction.

From Eq. (39), it is clear that $\lambda_m > \xi_m$. Now assume this is true for stage n, i.e., $\lambda_n > \xi_n$. Then, we also have $2\lambda_n > \xi_n > \xi_{n+1}$. Using this and Eq. (45) (the last term of which is easily shown to be positive), we obtain the following:

$$\begin{aligned}\lambda_{n+1} &> \frac{2\lambda_n(1+2\xi_{n+1})-\xi_{n+1}^2}{2(1+2\lambda_n)} = \frac{2\lambda_n+4\lambda_n\xi_{n+1}-\xi_{n+1}^2}{2(1+2\lambda_n)} \\ &> \frac{2\lambda_n+4\lambda_n\xi_{n+1}-(2\lambda_n)\xi_{n+1}}{2(1+2\lambda_n)} = \lambda_n\frac{1+\xi_{n+1}}{1+2\lambda_n} = \lambda_n\frac{\xi_{n+1}}{\xi_n}.\end{aligned}$$

From this, it follows that:

$$\frac{\lambda_{n+1}}{\xi_{n+1}} > \frac{\lambda_n}{\xi_n} > 1,$$

which proves that $\lambda_{n+1} > \xi_{n+1}$.

To prove $\lambda_{n+1} < \lambda_n$, we introduce the following variable:

$$\chi_n \equiv 2\lambda_n - \xi_n.$$

Using Eq.s (43) and (45), we have:

$$\begin{aligned}\lambda_{n+1} &= \frac{2\lambda_n(1+2\xi_{n+1})-\xi_{n+1}^2}{2(1+2\lambda_n)} + B_m\delta_n...\delta_m \\ &= \frac{2\lambda_n(1+\xi_{n+1})+\xi_{n+1}[(1+2\lambda_n)-(1+\xi_{n+1})]}{2(1+2\lambda_n)} + B_m\delta_n...\delta_m \\ &= \lambda_n\delta_n + \frac{\xi_{n+1}}{2}(1-\delta_n) + B_m\delta_n...\delta_m \\ &= \frac{\delta_n}{2}(2\lambda_n-\xi_{n+1}) + \frac{\xi_{n+1}}{2} + B_m\delta_n...\delta_m,\end{aligned}$$

where $B_m > 0$ is short for $[\lambda_m - \xi_m - 1/(2m-2)]/(m-1)$.

Thus,

$$\begin{aligned}2\lambda_{n+1}-\xi_{n+1} &= \delta_n(2\lambda_n-\xi_{n+1}) + 2B_m\delta_n...\delta_m, \\ &= \delta_n(2\lambda_n-\xi_n) + \delta_n(\xi_n-\xi_{n+1}) + 2B_m\delta_n...\delta_m,\end{aligned}$$

or:

$$\chi_{n+1} \;=\; \delta_n\chi_n + \delta_n(\xi_n - \xi_{n+1}) + 2B_m\delta_n...\delta_m,$$

From Eq.s (42) and (44),

$$\delta_n \;=\; \frac{\xi_{n+1}}{\xi_n} \;=\; \frac{1}{1+\chi_n} \;<\; 1.$$

Thus,

$$\chi_{n+1} \;=\; \frac{\chi_n}{1+\chi_n}(1+\xi_{n+1}) + 2B_m\delta_n...\delta_m,$$

To prove by induction, we can explicitly verify that $\chi_{m+1} < \chi_m$. Furthermore, if we make the induction assumption $\chi_n < \chi_{n-1}$, the above equation can be used to show:

$$\chi_{n+1} \;<\; \frac{\chi_{n-1}}{1+\chi_{n-1}}(1+\xi_n) + 2B_m\delta_{n-1}...\delta_m \;=\; \chi_n.$$

Having proved $\chi_{n+1} < \chi_n, \forall n$, we can rewrite it as:

$$2\lambda_{n+1} - \xi_{n+1} < 2\lambda_n - \xi_n,$$

or

$$2(\lambda_{n+1} - \lambda_n) \;<\; \xi_{n+1} - \xi_n \;<\; 0.$$

This completes the proof that $\lambda_{n+1} < \lambda_n$. ∎

F Proof of Proposition 5.2

The analysis of inter-dealer trading with a limit-order book under conditions of asymmetric information parallels previous analysis without informed trades.

First, assume that dealer L gets a positive quantity allocation from inter-dealer trading, then dealer W uses the following dominant strategy (the derivation is very similar to that in Appendix C):

$$x^W(p,z) = \begin{cases} \frac{\bar{v} - \xi\rho\tau_v^{-1}z - p}{\rho\tau_v^{-1}} & \text{if } z \in [\underline{s}, 1], \\ z & \text{if } z \in [0, \underline{s}), \end{cases} \tag{73}$$

where $\underline{s} \in [0,1]$ will be determined later.

Similar to the analysis in Appendix D, dealer i's ($i \neq W$) strategies are described by the following first-order condition:

$$\sum_{j\neq i}^{N} x_j'(p) = -\frac{[1-G(z)]/g(z)}{\bar{v} - \xi\rho\tau_v^{-1}z - p - \rho\tau_v^{-1}x_i(p)} = -\frac{1-z}{\bar{v} - \xi\rho\tau_v^{-1}z - p - \rho\tau_v^{-1}x_i(p)}.$$

Conjecturing a linear solution $x_i(p) = \mu - \gamma p$, then from the market clearing condition we can solve for:

$$z = \frac{\bar{v} - p + \rho\tau_v^{-1}(N-1)(\mu - \gamma p)}{\rho\tau_v^{-1}(1+\xi)}.$$

Substituting the above into dealer i's first-order condition:

$$\frac{1}{\sigma} = \frac{1 - \frac{\bar{v} - \xi\rho\tau_v^{-1}z - p}{\rho\tau_v^{-1}} - (N-1)(\mu - \gamma p)}{\bar{v} - \xi\rho\tau_v^{-1}z - p - \rho\tau_v^{-1}x_i},$$

where

$$\sigma = \frac{1}{(N-2)\gamma + \frac{1}{\rho\tau_v^{-1}}}.$$

Collecting terms to match the original conjecture, we have:

$$-\rho\tau_v^{-1}\gamma = (1 + \frac{\sigma}{\rho\tau_v^{-1}})\frac{\xi}{\xi+1}[1 + \rho\tau_v^{-1}(N-1)\gamma] - (N-1)\sigma\gamma - (1 + \frac{\sigma}{\rho\tau_v^{-1}}), \tag{74}$$

$$\rho\tau_v^{-1}\mu = (1 + \frac{\sigma}{\rho\tau_v^{-1}})\{\bar{v} - \frac{\xi}{\xi+1}[\bar{v} + \rho\tau_v^{-1}(N-1)\mu]\} - \sigma + \sigma(N-1)\mu.$$

Therefore, the solutions are:

$$\gamma = \frac{2(N-2) - N\xi + \sqrt{4N(N-2)(1+\xi) + N^2\xi^2}}{2(N-2)\rho\tau_v^{-1}(1+N\xi)},$$

$$\mu = \frac{(1 + \frac{\sigma}{\rho\tau_v^{-1}})\bar{v} - \sigma(1+\xi)}{\rho\tau_v^{-1}(1+\xi) + (N-1)(\rho\tau_v^{-1}\xi - \sigma)}.$$

The requirements that $z \geq x^W$ and $x_i \geq 0$, $\forall i \neq W$ are satisfied with the following choice of threshold customer order size:

$$\underline{s} = \frac{\gamma\bar{v} - \mu}{\rho\tau_v^{-1}(1+\xi)\gamma}.$$

Finally, we can verify that $\gamma > 0$. Thus, the second-order condition is always satisfied and the solution constitutes an equilibrium strategy. ∎

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Figure 1.

Equilibrium price vs customer order size in a dealership market. The solid line is for one-shot trading. The three dashed lines are for the initial stage of a two-stage trading model (the second lowest dealer inventory is 0.8, 1.0, 1.2 from top to bottom). The price for two-stage trading is generally higher than its one-shot counterpart at large customer order sizes, although at small customer order sizes it is influenced by dealer inventory.

Figure 1: Price vs Order Size in Dealership Market

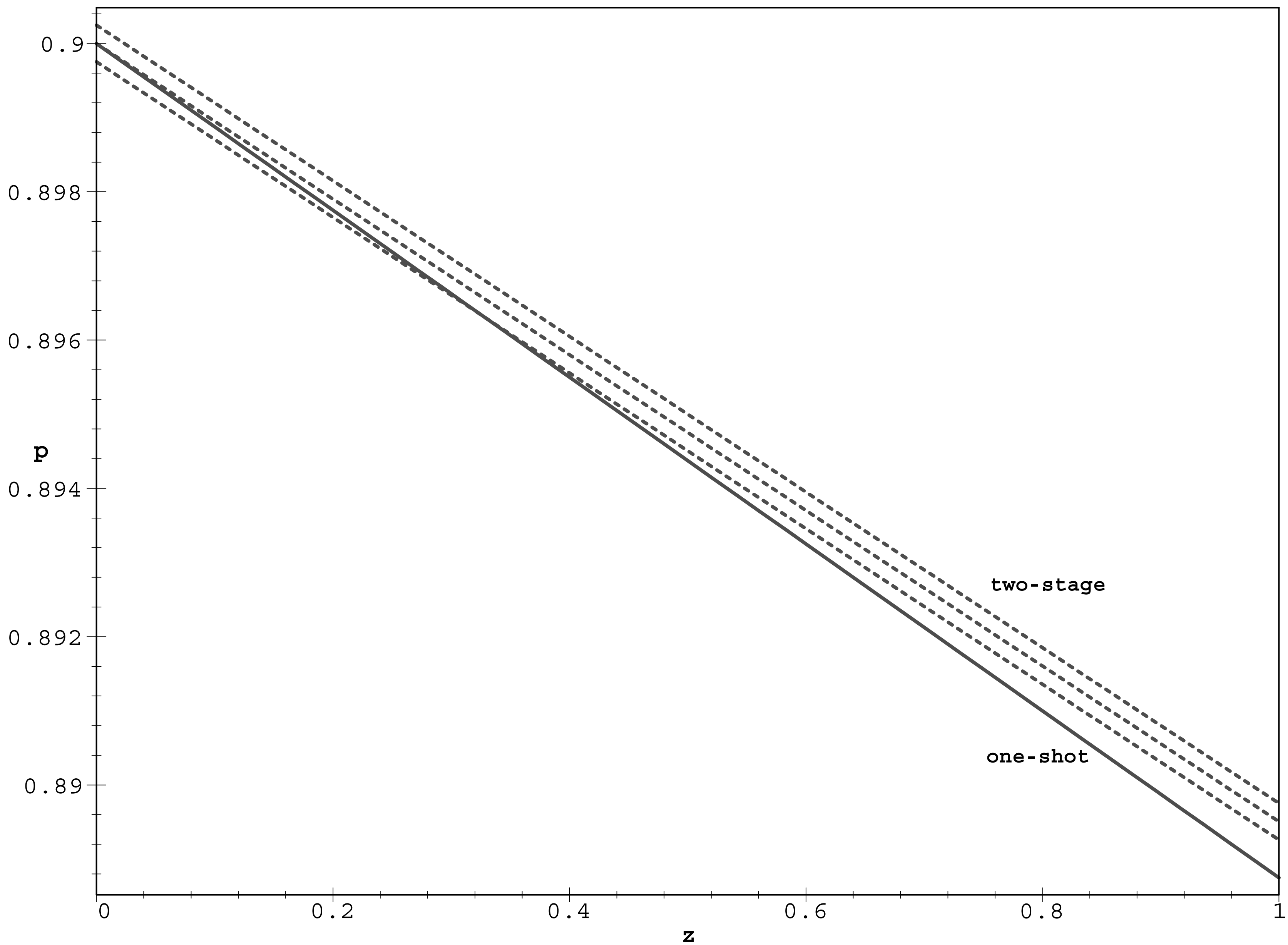

Figure 2.

Average price received by the customer vs customer order size in a limit-order book market. The solid line is for one-shot trading. The dashed line is for the initial stage of a two-stage trading model. All dealer inventories are set to be the same. The price for two-stage trading is generally higher than its one-shot counterpart at small customer order sizes, although at large customer order sizes it is lower.

Figure 2: Price vs Order Size in Limit-Order Book

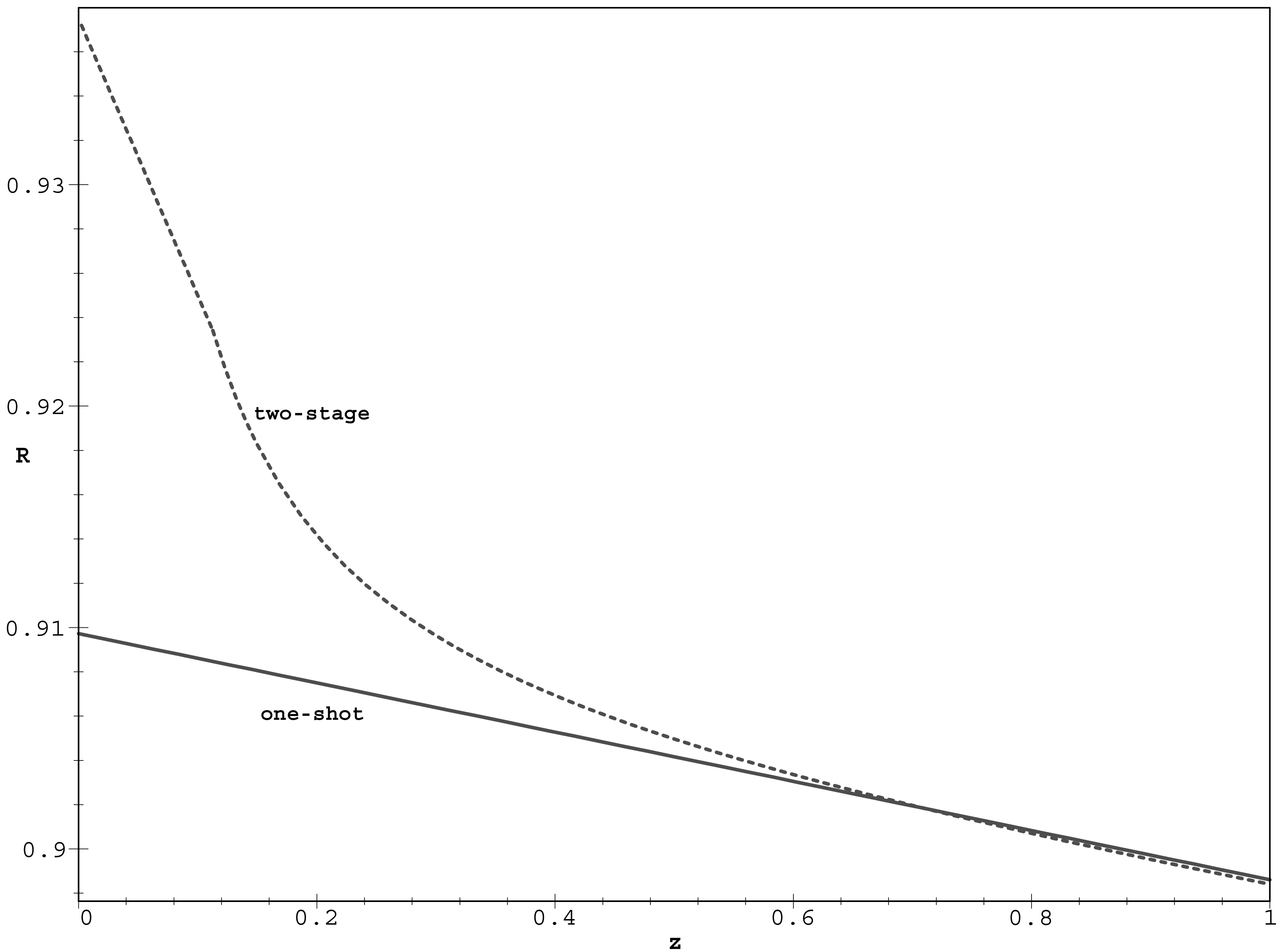

Figure 3.

The inverse liquidity parameter, λ, and inter-dealer trading volume vs the rounds of trading in a sequential inter-dealer auction market ($N = 12$). As trading progresses, the inter-dealer transaction volume as a percentage of the customer order is decreasing, and the market liquidity is also decreasing.

Figure 3: Liquidity and Volume in the Sequential Auction Inter-Dealer Market

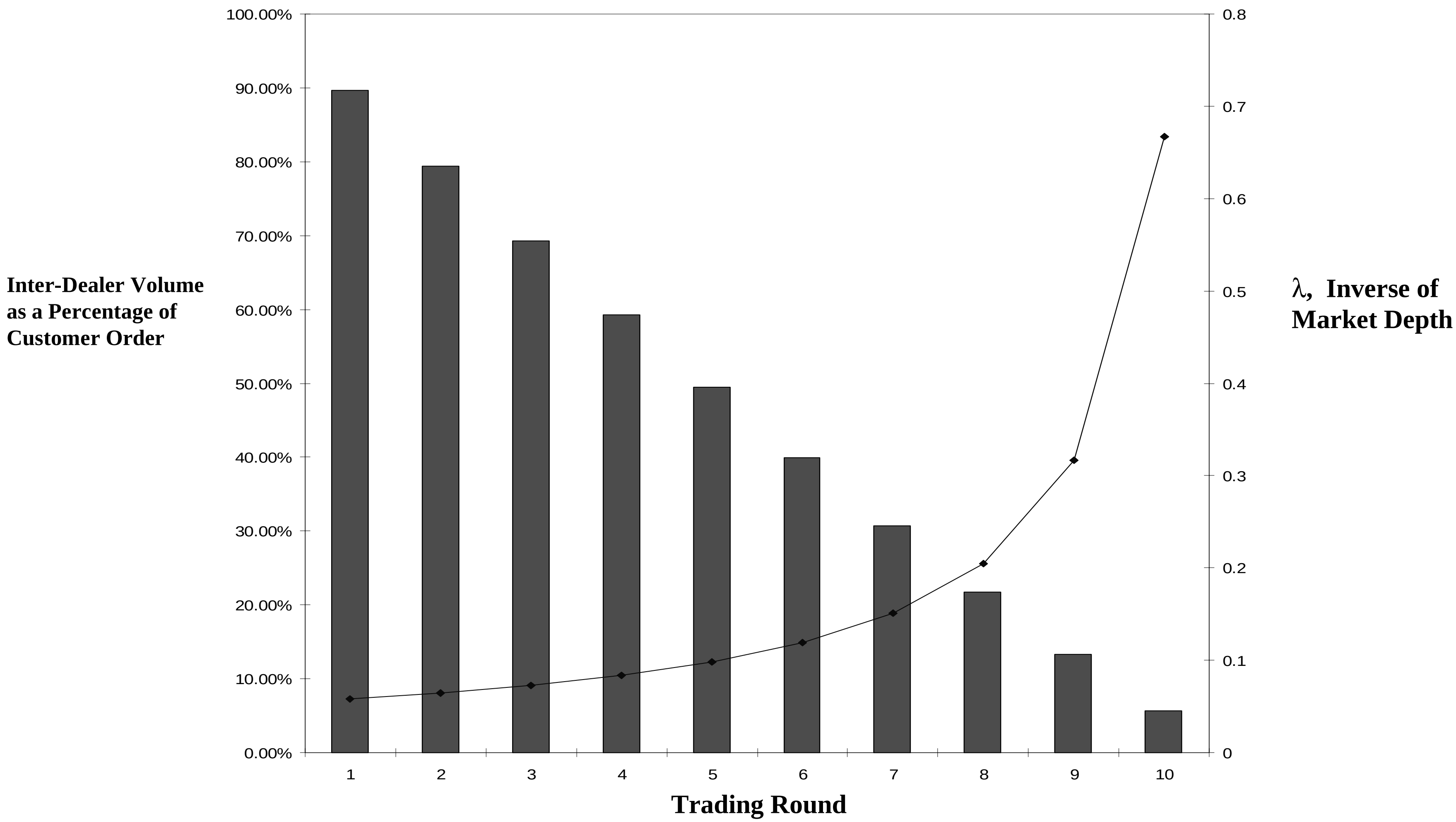

Figure 4.

Equilibrium price vs customer order size in a sequential inter-dealer auction market ($N = 10$). The solid line is the benchmark price for which there is no trading surplus for the dealers. The dashed lines are the dealers' demand curves with 1, 3, 5, and 7 rounds of trading remaining in the sequential game. With more rounds of inter-dealer trading, dealer competition is more intense and the customer's expected revenue is higher.

Figure 4: Price vs Order Size in a Sequential Game

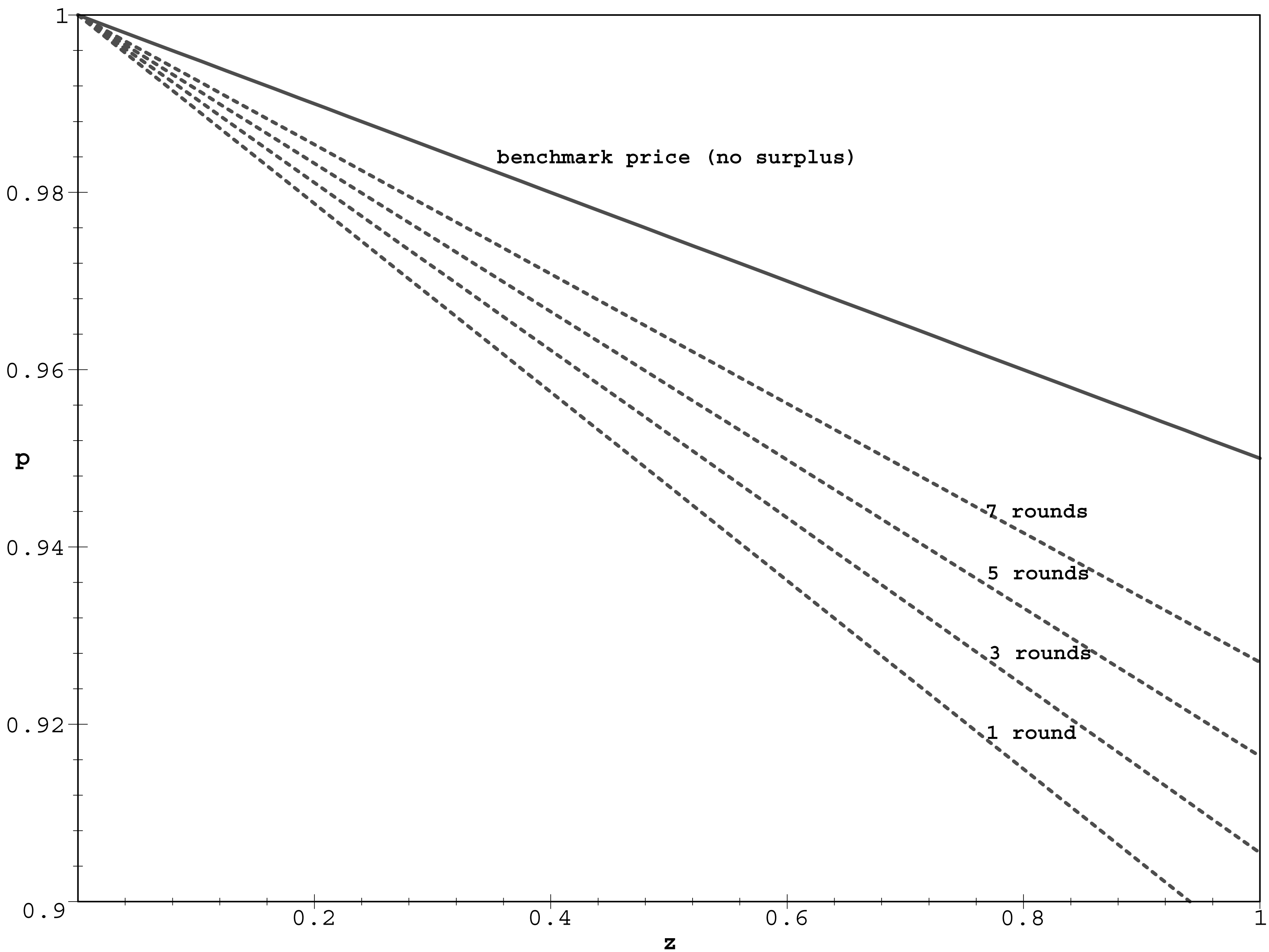

Figure 5.

The customer's expected revenue vs the total number of dealers in the absence of private customer information ($\xi = 0$). Except at very small number of dealers, the customer revenue is higher under an inter-dealer trading system with a sequence of auctions than with a limit-order book. We assume that sequential trading takes the maximum rounds of auctions (i.e., $m = 4$).

Figure 5: Revenue Comparison without Private Information

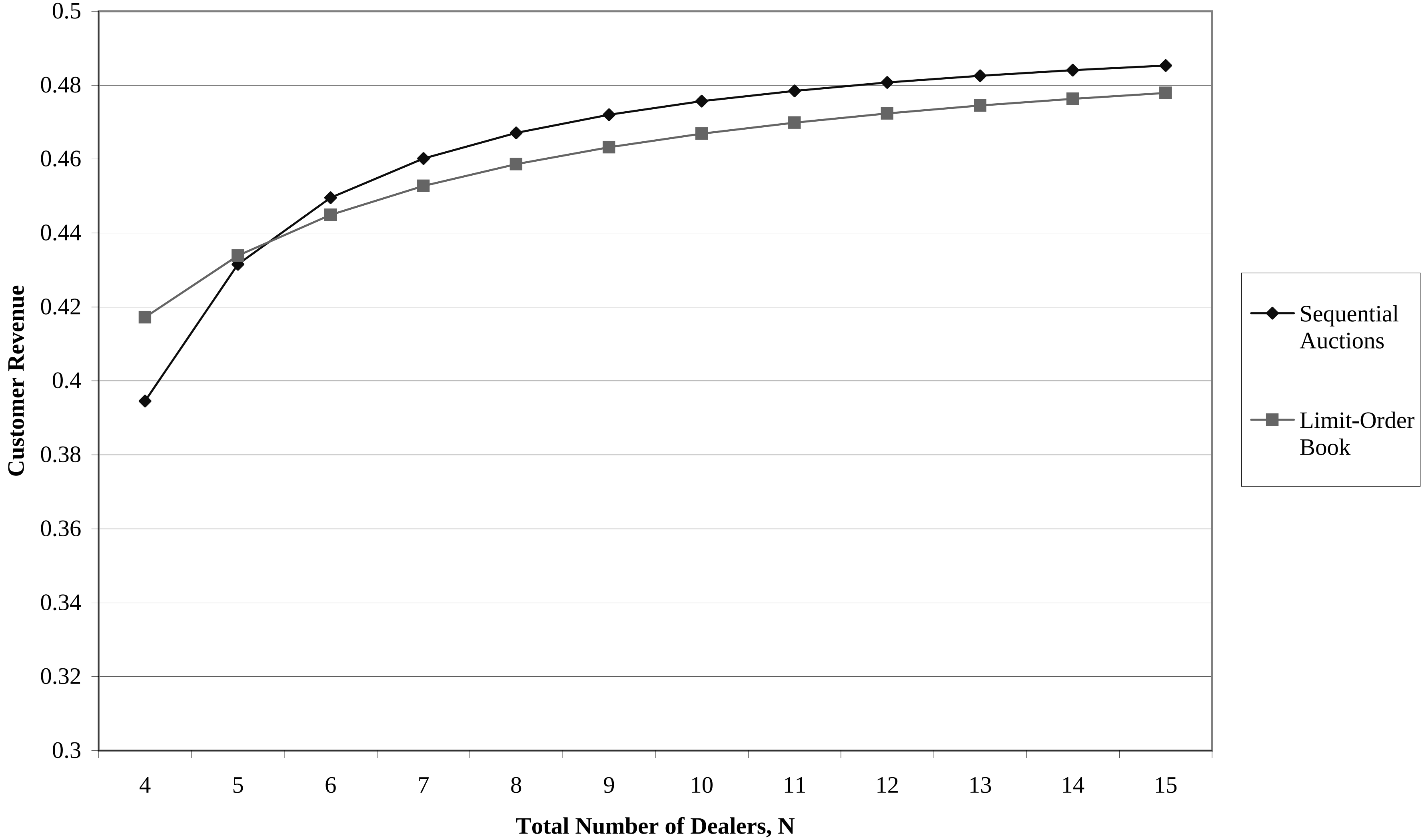

Figure 6.

The feasible and optimal numbers of trading rounds vs the information parameter, ξ, in a sequential auction inter-dealer market ($N = 15$). When there is no (or little) private customer information, there can be many rounds of inter-dealer trading. If the extent of private information (measured by ξ) is extremely large, inter-dealer trading will be made impossible (i.e., the number of feasible rounds of trading fall to zero, which is not shown in the figure). At intermediate values of ξ, the optimal number of rounds (from the customer's perspective) is smaller than the maximum feasible rounds of trading.

Figure 6: Number of Trading Rounds vs Private Information

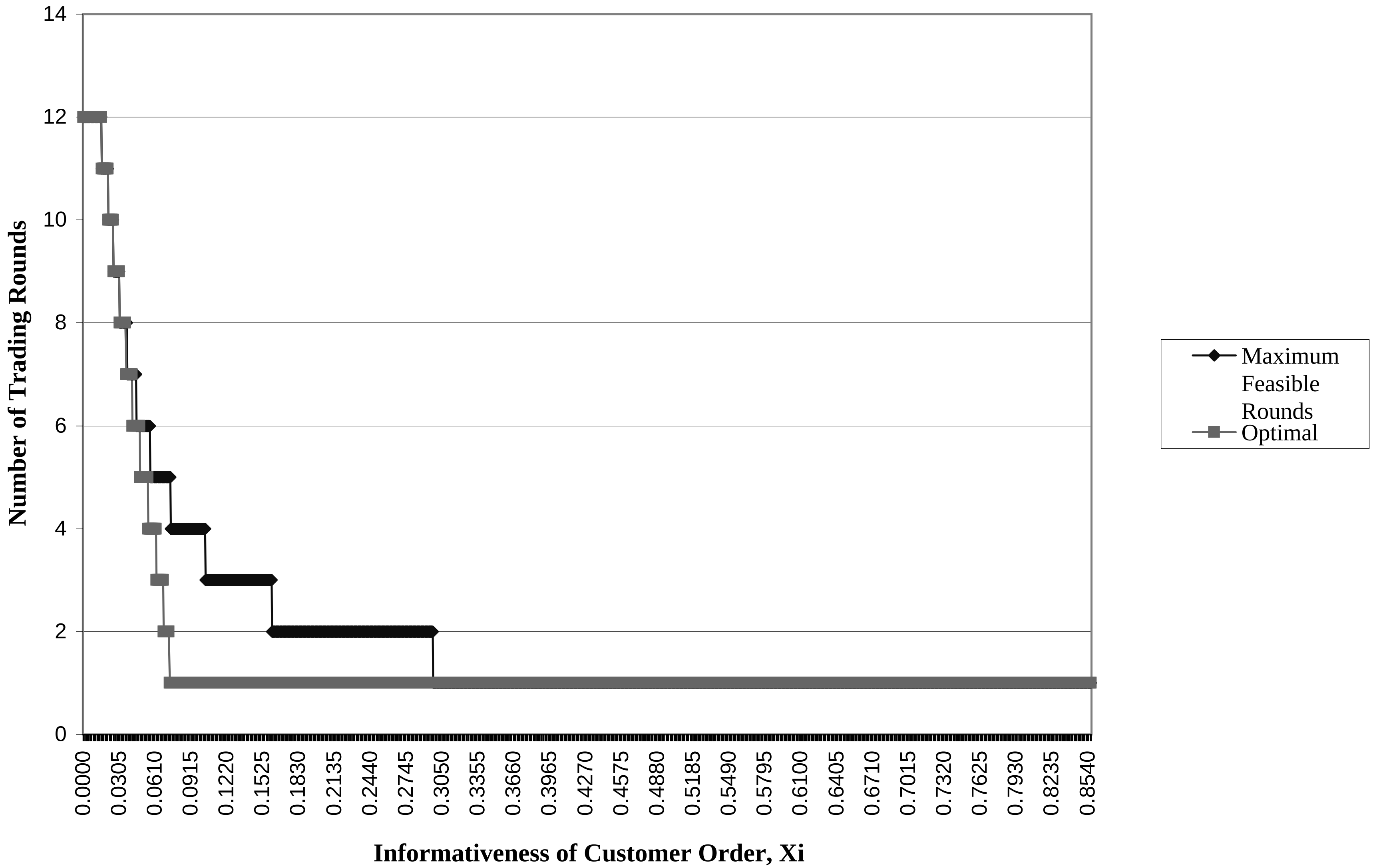

Figure 7.

The customer's expected revenue vs the information parameter, ξ. For the sequential inter-dealer market ($N = 15$), we assume that sequential trading takes the optimal rounds of auctions. At small ξ values, its customer revenue is higher than the customer revenue from an inter-dealer trading system based on a limit-order book. At higher values of ξ, however, the customer prefers limit-order book trading.

Figure 7: Revenue Comparison under Private Information