ANOTHER LOOK AT THE FORWARD-FUTURES PRICE DIFFERENTIAL IN LIBOR MARKETS

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The Eurodollar forward-futures price difference attributable to the daily settlement (or the Cox-Ingersoll-Ross proposition) has been estimated in the literature at from 2 to 48 basis points. These tests, however, ignore the fact that the CIR proposition requires convergence of spot and futures prices at expiration, which does not occur in the Eurodollar market given the expiration settlement feature of the futures. This paper examines this issue and provides new empirical tests with this problem removed. The results show that the differential is from 0 to 4 basis points and can be reduced even further with improved volatility estimates.

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In an important series of papers, Cox, Ingersoll, and Ross (1981), Richard and Sundaresan (1981), and Jarrow and Oldfield (1981) demonstrate that in the absence of default, a differential will likely exist between the price of a forward contract and the price of a futures contract with identical terms. This result is often referred to as the Cox-Ingersoll-Ross or CIR proposition. This differential would disappear only for contracts with one day remaining, or in a world of deterministic interest rates, or when interest rates and futures prices are uncorrelated. Thus, under most conditions a differential would likely exist.

While the assumptions required to establish a theoretical differential are relatively mild, whether the differential exists in practice is an empirical question. One of the most opportune markets for empirically testing the CIR proposition is the Eurodollar market. Eurodollar futures are among the most active contracts in the world, and the London Eurodollar rate, LIBOR, is widely used as the underlying in interest rate forwards, swaps, and options. Moreover, Eurodollar contracts should be uambiguously consistent with forward prices exceeding futures prices, because the correlation between futures prices and interest rates is clearly negative.[1]

Empirically testing the forward-futures differential is quite challenging. Futures price data are widely available, but forward price data are somewhat more difficult to obtain.[2] Even when forward price data are available, however, it can be difficult to obtain a sufficient quantity of observations. For example, many futures contracts have only four expirations a year. To compare the forward-futures price differential, it is necessary to match the price of a futures contract, which has a specific expiration, to that of a forward contract with the same expiration. Forward price data are often available only with certain expirations, which can make it difficult to obtain a sufficient

[1]Technically the condition is that futures prices are higher than forward prices if the covariance between futures prices and the price of zero coupon bonds is negative. In addition, futures contracts will have higher prices than forward contracts if the covariance between forward prices and zero coupon bond prices is negative or if the covariance between the spot price and the zero coupon bond price is less than the variance of the zero coupon bond price.

[2]At this point, we should distinguish between actual forward contracts and synthetic forward contracts. An actual forward contract price would be the transaction price for a forward contract directly entered into between two parties. Such data are virtually never available because the transactions are private. A synthetic long (short) forward contract can be constructed by purchasing (selling) the underlying asset and borrowing (lending). The synthetic forward price and actual forward price should be the same or at least, extremely close, or otherwise there would be an opportunity for an arbitrage profit. Synthetic forward prices are, therefore, constructed from spot prices and interest rates and referred to as forward prices.

number of observations that align with the futures contract expirations.[3] Moreover, measurement errors in the data can greatly interfere with the ability to draw reliable inferences from such tests.[4] Indeed, research on the forward-futures price differential is fraught with data difficulties and inconsistencies, possibly explaining why there has been so little research on this subject. Published empirical tests to date, to be discussed in more detail later, document the existence of a significant differential, ranging from 2 to 48 basis points, varying expiration.

This paper shows that these empirical tests are biased, because they fail to recognize a requirement of the CIR proposition: the convergence of spot and futures prices at expiration. This convergence does not occur in the Eurodollar market. The CIR proposition addresses the effect of the different cash flow streams, which result from the daily settlement, on the difference between forward and futures prices. Proof of the CIR proposition requires the ability to buy and hold an underlying asset and convergence of the price of that asset to the futures price at expiration. In Eurodollar markets, the underlying asset is a Eurodollar time deposit, a dollar loan made from one London bank to another. Interest on these loans is based on the add-on method, whereas the Eurodollar futures contract is based on the discount method. In this paper, we show that this problem results in non-convergence of spot and futures prices at expiration, which carries backward into a price differential prior to expiration that is totally unrelated to the daily settlement, or CIR, effect. Thus, the results in the literature that purport to test the CIR proposition are corrupted. A formal derivation of the CIR proposition for Eurodollar contracts is in Appendix A.

We refer to this difference in how the spot and futures settle at expiration as the *expiration settlement effect*. Our interest is in the Cox-Ingersoll-Ross proposition, which we refer to as the *daily settlement effect*. This paper provides an examination of the impact of the expiration settlement effect on the daily settlement effect, a discussion of empirical tests of the effect of the expiration settlement on the daily settlement, and an adjusted empirical test that blends market data with numerical results that have been cleansed of the expiration settlement effect. Using a simple volatility estimation model,

[3]Consider the following example, which would apply to many interest rate contracts. A given futures contract has expirations of a specific day in March, June, September, and December. The underlying has a 90-day maturity at the futures expiration. Interest rate forward contracts typically are offered with maturities of 30, 60, 90, …, 360 days. In a given year, the March futures contract price can be matched with the price of a forward contract two times (e.g., in January 60 days ahead and in February 30 days ahead), the June contract can be matched five times, the September contract can be matched eight times, and the December contract can be matched eleven times. Thus if one has 250 days of observations in a year, only 26 are usable. Some of these dates will fall on weekends so even fewer might truly be available. Thus, a researcher may often have to discard over 90 percent of the data.

[4]See, for example, French's (1983) study of the copper and silver markets.

our results show that the forward-futures price differential is more likely to be anywhere from zero to four basis points. With a better volatility estimation model, the results appear to be less than one basis point. This finding is important, because it shows that in one of the most active forward and futures markets of the world, the impact of the difference in cash flow streams is extremely small, a conclusion that might not be drawn from the existing literature.

Section I examines the two previous empirical studies on this subject and discusses the problems involved in empirically testing the CIR proposition. Section II describes the data and the design of the empirical tests that will be done in this study. Section III reports the results of empirical tests that provide a simple comparison of actual forward and futures prices. Section IV describes how we fit an arbitrage-free binomial tree to forward market data and use that tree to produce arbitrage-free futures prices that will be free of the expiration settlement effect. Section V reports the results of tests of these hypothetical futures prices in comparison to forward prices. Section VI provides a summary and conclusions.

I. Previous Studies and the Problems of Testing the Forward-Futures Price Differential

Futures contracts on 90-day Eurodollars at the Chicago Mercantile Exchange is the most active futures contract in the U. S., with 2003 volume of over 202 million contracts, with each contract covering $1 million face value. Open interest at year-end 2003 was over 4.8 million contracts, representing about $4.8 trillion of Eurodollars. The Eurodollar forward market is also one of the most active financial markets. The Bank for International Settlements (www.bis.org) reports that as of June 2003 (latest data available), the global market for interest rate forward contracts, called FRAs, is about $10.3 trillion in notional principal with a market value of $20 billion. U. S. dollar transactions are about 44 percent of global over-the-counter derivatives, suggesting that notional principal in the FRA market is about $4.5 trillion. One of the reasons for the success of the Eurodollar futures contract is the fact that over-the-counter derivatives dealers use the contract to hedge their positions in LIBOR-based interest rate swaps, forwards, and options. Indeed, this market is large, highly successful, and probably the most opportune market in which to examine the question of whether forward prices equal futures prices.

A. Prior Research

The first empirical study of the forward-futures price differential in LIBOR markets is Meulbroek (1992), whose data consist of daily LIBOR rates for 1-, 2-, 3-, 6-, and 12-month maturities for the period March 1982 to June 1987, a total of 1,232 days.

Because of the aforementioned limited availability of data, she infers forward rates for other maturities by fitting a quadratic regression. There is some uncertainty, however, in the manner in which Meulbroek calculates rates. Her equation (1) is correct for the calculation of Eurodollar forward rates, but there is no indication of how she calculates futures rates. Assume, as an example, that on the day of expiration of a futures contract 90-day spot LIBOR is 6%. Price reporting services will state that the contract settled at a price of 100 – 6 = 94. The actual futures price, however, is 100 - 6(90/360) = 98.5. Thus, the terms of the contract specify that the buyer is effectively purchasing 100 Eurodollars at a price of 98.5. This result can imply a compound rate of $(100/98.5)^{365/90} - 1 = .063212$, or $(100/98.5)^{360/90} - 1 = .062319$ if a 360-day year is assumed. Since the spot market for Eurodollars calculates interest in the add-on manner, however, the rate could be found differently. A Eurodollar time deposit of 100 would grow to 100(1 + .06(90/360)) = 101.50 for an effective rate of $(101.50/100)^{365/90} - 1 = .062242$ or $(101.50/100)^{360/90} - 1 = .061364$ using 360 days. Thus, the "futures rate" could be considered to be 6%, 6.3212%, 6.2319%, 6.2242%, or 6.1364%. Except for the third and fourth numbers, the variation in basis points is substantial. It is not even clear which is the correct or best number.

Meulbroek fits the term structure based on LIBOR spot rates, which are handled correctly and unambiguously, but some of the results (e.g., her Table III, p. 388) discuss the statistical properties of forward and futures rates so the issue of how the futures rates are defined is important. Fortunately, most of her remaining results are in terms of prices so there is no ambiguity. She tests several related questions, but the primary one of our interest is the sign and magnitude of the forward-futures price differential. As noted, the Cox, Ingersoll, Ross propositions state that futures prices will exceed corresponding forward prices if the correlation between futures prices and interest rates is positive.

Meulbroek measures the futures price minus the forward price, so she should obtain negative values. Her results show for the nearby contract that the average differential is –0.0002 for a contract par value of $1. For the second nearby contract, the differential is –0.0024. For the third nearby contract, the differential is –0.0048. As we show later, however, it would be important to know if any of the sample observations are positive, because there is reason to believe that some, if not many, probably are, as we show later.

Grinblatt and Jegadeesh (1996) examine daily data over the period 1982-1992 for spot maturities of 1-, 2-, 3-, 6-, and 12-months. Because essentially all of their results are in terms of rate differences instead of price differences, the question of how the futures rate is defined is more severe for this paper. While they show that they correctly relate Eurodollar spot prices to Eurodollar spot rates, they state that F(s,τ;t) is the "Annualized Eurodollar futures rate at t for the interval s to τ. (t is omitted when t = 0.)" (p. 1501). They proceed (p. 1502) to state that the futures rate is given as $F(s,s+0.25;t) = 1 - \text{Futures Price}_t/100$ (their equation (2)). As in the example we gave above, this would suggest that 6% is the rate they would use. As we stated above, it is not clear that this is the correct value for comparison purposes. Hence, an argument could be strongly made to avoid rate comparisons at all costs. Price comparisons are clearly preferred, and the CIR propositions are stated in terms of price comparisons.

To generate a sufficient number of matched observations, Grinblatt and Jegadeesh proceed to fit two interpolated term structures. One term structure fits a cubic spline to the available futures rates. Of course what one obtains from this procedure would be unclear, given the aforementioned problem in defining the appropriate "futures rate." The other method fits a cubic spline to the spot LIBOR rates, which would be clearly appropriate and consistent with Meulbroek. All of their remaining results are based on rate differences obtained from these estimated term structures. Half of the tests are based on a comparison of LIBOR forward rates compared to date-matched futures rates obtained from the interpolated futures term structure, and half are based on a comparison of futures rates compared to date-matched forward rates obtained from the interpolated term structure. In all cases, however, rates are compared to rates.

Their forward rate is defined as [(1/forward price) − 1]360/90, which is correct for the LIBOR market. Again, their futures rate is 1 − futures price. From these results, we would like to be able to convert their rate differences into price differences. Unfortunately, algebraic rearrangement of these expressions indicates that this is not possible without knowing the level of the forward rate. Using the data set we describe later in this paper, we estimate the average forward rate for a period that overlaps the Grinblatt-Jegadeesh study and the present study, 1987-1992. Using this information with their quoted results, we find that the price differences in basis points per $1 par are estimated at

Maturity	Average	Range
(using the futures term structure)		
0 – 3 months	-3.2	-0.5 to –7.2
3 - 6 months	-3.2	-0.7 to –7.5

6 - 9 months	-6.2	-3.6 to –10.2
(using the spot/forward LIBOR term structure)		
0 – 3 months	-2.3	-0.2 to 6.5
3 - 6 months	-3.2	-0.7 to –7.5
6 - 9 months	-4.6	-0.2 to –8.7

These differences are smaller than those reported by Meulbroek, but hers are exact and these are rough estimates. As we noted, Meulbroek did not report maximum and minimum values and gave no indication if any differences were positive, but Grinblatt and Jegadeesh's results do show some negative rate differences, which lead to positive price differences (0-3 months in the spot/forward LIBOR term structure above). These negative rate differences are particularly noticeable in the graphs in their Figure 1 (pp. 1504-1506). Such differences suggest that investors are irrational and do not have a preference for the settlement at expiration of forward contracts over the daily settlement of futures contracts, as Cox-Ingersoll-Ross prove they would. More importantly, these negative rate differences that lead to positive price differences bias the overall average toward zero, leading to a downward bias when attempting to ascertain the true differences between futures and forward prices or rates.

A second stage of the Grinblatt-Jegadeesh paper involves the estimation of theoretical differences in forward and futures rates using the Cox-Ingersoll-Ross (1985) and Vasicek (1977) term structure models. These theoretical differences are compared to actual differences. Given the large gap between the two, Grinblatt and Jegadeesh search for alternative explanations in the form of market imperfections or simple mispricing of the futures. They conclude that the difference is due to an unexplained lack of arbitrage activity. Their data set is used to estimate the parameters of both models, but an important limitation of both of these models is not addressed in their study. It is well known that a model like the Cox-Ingersoll-Ross or Vasicek model is arbitrage-free only within the model itself but not external to the model. Both models produce the term structure as an output. Neither model is calibrated to the current term structure. Hence, anyone trading in an actual market with one of these models is susceptible to arbitrage losses to other parties. Fully arbitrage-free models like Heath-Jarrow-Morton (1992), Ho-Lee (1986), or Black-Derman-Toy (1990) fit the current term structure and prohibit all arbitrage in the market and would have been a better choice.

B. Complications from the Expiration Settlement Feature of Eurodollar Futures

An even greater concern and a central theme of the present paper is that the tests of Meulbroek and Grinblatt-Jegadeesh do not account for a critical characteristic of the Eurodollar futures market. Let F(t,T) be the price on day t of a futures contract expiring at time T. Now move ahead to time T, the expiration. A 90-day Eurodollar time deposit, the instrument on which the futures contract is based, has a rate of L(T,T+90). At expiration, the present value of $1 ninety days later is

$$B(T,T+90) = \frac{1}{1 + L(T,T+90)(90/360)}.$$

This value can be viewed as the spot price at expiration. The futures price at expiration is defined by the terms specified in the contract by the Chicago Mercantile Exchange:

$$F(T,T) = 1 - L(T,T+90)(90/360).$$

Note that the spot instrument, the rate of which drives the futures price, is in the form of an instrument in which the interest is added on to the amount invested. That is, B(T,T+90) dollars invested in a Eurodollar time deposit grows to $1 by the factor 1 + L(T,T+90)(90/360). The futures contract is priced, however, as though the underlying were a discount instrument, such as a U. S. Treasury bill.[5]

Standard cost-of-carry theory for futures prices typically ignores the daily settlement feature and proposes a hypothetical transaction consisting of buying the spot instrument and hedging its future delivery by selling a futures contract. Consider the following strategy:

Time t

Buy a Eurodollar time deposit that pays $1 at time T + 90 by investing B(t,T+90).

Sell a Eurodollar futures expiring at time T for price F(t,T)

Time T

The Eurodollar time deposit is worth

$$B(T,T+90) = \frac{1}{1 + L(T,T+90)(90/360)}.$$

The futures price is

$$F(T,T) = 1 - L(T,T+90)(90/360).$$

The payoff of the futures is F(t,T) – F(T,T) or

$$F(t,T) - (1 - L(T,T+90)(90/360)).$$

[5]It appears as if the Eurodollar contract design was cloned from the U. S. Treasury bill futures contract, which had begun trading six years earlier. The Treasury bill contract was successful and traders were familiar with it. Moreover, this particular design permits a contract with $1 million face value to change in value by $25, given a one basis point move in the underlying rate. This feature facilitates the mental effort required to trade quickly. Ironically, the Treasury bill contract is no longer very actively traded, while the Eurodollar contract is the most active U. S. futures contract. In fact, in the year 2003 for every Treasury bill futures contract that trades, over 40,000 Eurodollar contracts trade.

The value of the overall position of the futures contract and the Eurodollar time deposit is

$$\frac{1}{1+L(T,T+90)(90/360)} + F(t,T) - (1 - L(T,T+90)(90/360)).$$

To derive an arbitrage-free futures price, this transaction, which is typically called cash-and-carry, must be risk-free. In this case, however, it is not. In simple terms, the futures price and spot price do not converge. Moreover, there is no way to alter one's holdings of either the Eurodollar time deposit or Eurodollar futures to offset the risk.[6] Hence, it is not possible to price the futures contract using this approach. If the instrument were a forward contract, convergence would occur and lead to the standard cost of carry pricing formula. The convergence of futures and spot prices is a common feature of most futures contracts and plays a critical role in the Cox, Ingersoll, and Ross propositions. The effect of this non-convergence of futures and spot prices is not considered in the Meulbroek and Grinblatt-Jegadeesh tests.

This unusual settlement feature of the Eurodollar futures contract has, however, been discussed by Sundaresan (1991), who provides an adjusted model based on the general equilibrium term structure model of Cox, Ingersoll, and Ross (1985).[7] His adjustment accounts for the feature, but requires a numerical solution to a partial differential equation. Sundaresan computes forward-futures price differentials based on a range of assumed inputs and finds an average differential of less than -0.0002 per $1 par for contracts expiring in 90 days and -0.0006 for contracts expiring in 180 days.[8] These differences are, however, not based on actual data, but on assumptions about the parameters of the stochastic process driving the spot rate as well as an assumed level of the spot rate. Sundaresan does provide some limited empirical evidence based on 13

[6]The problem boils down to the simple equation, 1/(1 + y) – (1 + y) = 0. There is no solution except y = 0, which would imply an interest rate of zero. Alternatively, if a weight is applied to one term, the solution for the weight is a function of the random interest rate. Thus, for a simple cash-and-carry transaction (buy spot, sell futures), there is no ex ante perfect hedge. It is, however, still possible to price the futures. As we discuss later, any futures contract is the expectation of the futures price at a later date, where expectations are taken using the martingale probability measure. We shall do this within a hypothetical setting using the Heath-Jarrow-Morton (1992) model, but it can be done without resorting to any particular model of the term structure, provided that the term structure is arbitrage-free.

[7]Even though it was published in 1991, Meulbroek (1992) does not cite the Sundaresan paper. Grinblatt and Jedgadeesh (1996) cite it but make only a casual reference to it as having addressed the pricing of Eurodollar contracts. They do not discuss the primary theme of the Sundaresan paper, the complexity added by the settlement feature.

[8]Some adjustments are required to interpret Sundaresan's findings. Consider his Table 2 on p. 419. For the first case, the futures price is given as 92.74 and the forward price is given as 92.78. These are quoted prices, which are simple linear prices, based on a subtraction of a rate from a par value of 100. The actual futures price would be found as 100 – (100 – 92.74)(90/360) = 98.185. The forward price would be equivalent to 100 – (100 – 92.78)(90/360) = 98.195. This is a difference of 0.01. Using $1 par as standard, the difference is 0.0001. Sundaresan quotes the difference as 4.66 basis points, which is based on the quoted prices of 92.74 and 92.78, carried out to more decimal places, and multiplied by 100. To adjust for this effect, the results in the final column in Sundaresan's Tables 2 and 3 should be multiplied by a factor of 1/(4*10,000).

observations, another clear case of the aforementioned data availability problem, which are also subject to the settlement problem. He finds that the difference averages $0.00125 per $1 par for 90-day contracts.[9]

Thus, Meulbroek's results, using empirical data over a five-year period, show a differential of -0.0002 to -0.0048 per $1. Grinblatt and Jegadeesh, using empirical data on rate differences over a 10-year period (but converted to price differences over a five-year period) show differences of –0.00002 to –0.0010 per $1 with some results being irrationally positive. Sundaresan, using the Cox, Ingersoll, and Ross model and hypothetical inputs, shows a differential of -0.0002 to -0.0006, and -0.00125 using empirical data.[10] Sundaresan's work, which accounts for the settlement difference, is based on numerical inputs into a model that does not fit the current term structure. His limited empirical results are also subject to the expiration settlement problem. There is clearly a need for additional research that uses a sufficient amount of quality data and addresses the expiration settlement problem before we can conclude that the magnitude of the daily settlement is either large or small.

For a rough idea of the magnitude of the expiration settlement effect, Figure 1 shows the difference between forward and futures prices at expiration for a $1 par contract for a reasonable range of LIBOR values of 1 to 10 %. For example, if LIBOR is 6%, the forward price is 1/(1 + .06(90/360)) = 0.98522167, the futures price is 1 - .06(90/360) = 0.9850, and the difference is 0.00022167 or 2.2167 basis points. We see that the price differential is positively related to the magnitude of LIBOR and increases at an increasing rate with LIBOR. The settlement effect accounts for up to 10 basis points of the price differential at expiration, which is equivalent to $1,000 for a standard $1 million Eurodollar futures contract. This amount is all the more significant considering that the initial and maintenance margins on the futures contract are less than $1,000.

While Figure 1 shows the impact of the expiration settlement effect without considering the effect of the daily settlement, our interest is in the impact of the daily settlement without the expiration settlement. We have reason to believe that not

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[9]The forward-futures price differential has also been studied in other markets, some of which are interest-rate instruments (usually Treasury bills) and some of which are other types of assets. See Elton, Gruber, and Rentzler (1984) Kolb and Gay (1985), Capozza and Cornell (1979), Cornell and Reinganum (1981), Kawaller and Koch (1984), Gendreau (1985), Park and Chen (1985), and Allen and Thurston (1988). In fact, virtually any study of the pricing of futures contracts in relation to spot prices treats the futures contract as though it were a forward contract and is, therefore, a joint test of pricing efficiency and the forward-futures price differential.

[10]Technically, Meulbroek's differences are slightly less negative than stated in comparison to Sundaresan's because she expresses her difference as the log of the futures price over the forward price. For negative differences, the log difference will be slightly more negative. This effect is, however, fairly minor.

accounting for the expiration settlement effect could substantially bias any conclusions we might draw on the difference between forward and futures prices that is driven by the different daily cash flow patterns in the two contracts. Previous research has either failed to account for the expiration settlement effect or has done only a simple analysis using arbitrary inputs chosen by the researcher. Clearly there is a need to analyze the issue using market data, while accounting for the expiration settlement effect.

Some insights into this problem have been examined under an alternative guise. The difference between forward and futures prices is oftentimes attributed to an effect known as *convexity*. Burghardt and Hoskins (1995a, 1995) and McDonald (2003) describe the convexity effect as arising out of a timing difference in a hedge transaction. It is well-known that if a spot transaction can be perfectly hedged, then the overall position should earn the risk-free rate. To prevent arbitrage, the futures price must equal the spot price increased by the carrying cost and decreased by any known cash flows on the spot asset. This formulation of the futures price is commonly referred to as the cost of carry model. A perfect hedge can be constructed using a forward contract, so the cost of carry model unquestionably applies to forward contracts. If a perfect hedge cannot be constructed using a futures contract, the cost of carry model would not apply to futures contracts, thereby implying that forward and futures prices would almost surely differ.

In Appendix B, we show that this convexity effect is strictly a product of the fact that the futures contract is designed to pay off in a different manner from that of the underlying spot instrument. We show that if the spot instrument is a discount instrument, and the futures is designed as an add-on instrument, or vice versa, a perfect hedge is not possible. If both instruments are add-on or both are discount, however, a perfect hedge is possible. The notion of convexity arises, however, not from a "timing" problem, but rather from the failure of futures and spot prices to converge at expiration. That this is not a "timing" problem is easily seen by the fact that a hedge put in place an instant before expiration would suffer from the same problem. The idea that the mathematical concept of "convexity" applies comes from the fact that the payoff of a discount instrument is a linear function of the interest rate (negative first and zero second derivatives with respect to the interest rate), while the payoff of an add-on instrument is a non-linear function (with negative first and second derivatives) of the interest rate. When both instruments are not priced in the same manner, one instrument gains an "advantage" over the other, and a perfect hedge is not possible.

McDonald (p. 215) and Gupta and Subrahmanyam (2000) mention the daily settlement as a source of the convexity bias, but it should be clear that a price difference arising from the daily settlement would still exist if both spot and futures instruments settle in the same manner at expiration. Likewise, a price difference arising from the expiration settlement would exist if there is no daily settlement. It is important not to confound these two effects.

This study re-examines the issue of the forward-futures price differential arising from the daily settlement in two ways. Using a new and larger empirical data set, we examine the actual differences, showing how real data can lead to misleading and inconsistent results. We then examine the differences using estimated futures prices that are based on an arbitrage-free model of the term structure. In doing so, we control for the expiration settlement feature of Eurodollar futures.

II. Data and Test Design

The tests reported here use data from the London Eurodollar market. The British Bankers Association (BBA) samples a set of 16 London banks and establishes an official LIBOR at 11:00 a.m. London time each business day. This rate is based on the interquartile range of quotes of the sample banks. The BBA publishes the data on a daily basis on its web site www.bba.org.uk. The data set begins in 1987 and consists of spot LIBOR for Eurodollar deposits of one to 12 months in increments of one month.[11] Thus, the data are quite granular, allowing a fit of a fairly detailed term structure of up to one year, for every day since 1987. For this study, we use data over the period of 1987 through 2000, a total of over 3,500 term structures, so this study covers a longer time period than other studies.

We obtain data on the Eurodollar futures contract of the Chicago Mercantile Exchange from the Institute for Financial Markets, the designated supplier of CME data. From its beginning in 1982 through 1994, the contract was available with expirations in March, June, September, and December. Starting in 1995, a November expiration was added, and the following year, expirations of each month became available.[12] These expirations go out a different number of years, depending on the year in the period 1982-2000. In recent years, these have gone out as far as ten years. Given the limitations of the forward market data, however, we are interested in expirations of less than one year.

[11]The Eurodollar market assumes 30 days in a month when quoting rates, but interest calculations typically use the actual number of days.

[12]Technically, the expirations available for trading are March, June, September, and December, plus the current month and the next month. Therefore, there will be a contract expiring each month of the year from 1996 onward. The exact expiration day is the second London business day prior to the third Wednesday of the month.

The availability of these monthly expirations, however, provides a greater number of observations than in previous studies, helping to address one of the problems of earlier empirical studies of this type.

The empirical component of this study compares forward prices from the BBA LIBOR data with futures prices from the CME. This examination requires a careful alignment of contracts and dates, which is discussed in the next section. One other concern, however, is the potential for non-synchronous data. The BBA LIBOR rates are established at 11:00 a.m. London time. The futures prices are settlement prices as of the close. This difference in time can be problematic. Interestingly, however, the actual spot market for Eurodollars in London is itself not synchronized with the futures market in Chicago. The BBA establishes the official daily LIBOR around 11:00 London time and publishes it soon thereafter. The futures contract begins trading in Chicago at 7:20 a.m. central time and closes at 2:00 p.m. The official LIBOR is not updated for another day, though unofficial quotes are available from various services. Nonetheless, the Chicago futures market continues to trade until well after the London banks have closed. Thus, in reality the two markets themselves are not synchronized, which can cause further problems in trying to test the relationship between forward and futures prices.[13]

Another concern is that futures prices are affected by transaction costs, taxes, and bid-ask spreads that differ from those of forward prices extracted from the LIBOR term structure. They, too, are affected by these factors, but to a different extent.[14]

Finally, we should note that a test of this sort using futures price data would suffer from the same problems of the Meulbroek and Grinblatt-Jegadeesh studies. That is, the futures contract is based on the CME's design of the futures contract as a discount instrument, while the underlying Eurodollar is an add-on instrument. We shall attempt to address this problem by adjusting the forward prices so that they reflect the prices of instruments constructed from the same rates, but based on the discount procedure used in the futures market.

While the empirical study is conducted to determine if a different, perhaps better, data set can resolve the question, the expiration settlement feature will still corrupt our results. Thus, we need an alternative approach free of both the expiration

[13]It is not clear whether Meulbroek and Grinblatt-Jegadeesh had synchronous data. Meulbroek states that the futures prices were obtained from Goldman Sachs, which were evidently end-of-day prices. Her spot rates were the "latest available from Telerate at the close of the futures market." Thus, they seem relatively synchronized. Grinblatt and Jegadeesh obtain their futures price data from the CME and the spot data from DRI. There is no indication of whether they are synchronized. It is likely they are not, since the DRI data were stated as being averages from several banks, obtained from Reuters. In all likelihood, these were the 11:00 a.m. quotes.

[14]Grinblatt and Jegadeesh address some of these market imperfections and conclude that they are insufficient to affect their observed rate differentials to any great extent.

settlement issue and non-synchroneity problem. This study estimates input parameters from empirical data and fits the term structure to the more general Heath, Jarrow, Morton (1992) model. The Heath-Jarrow-Morton (HJM) model has the advantage of requiring the estimation of only one general parameter, the term structure of volatility. In addition, the HJM model is arbitrage-free in that it fits the current term structure of interest rates. The Cox-Ingersoll-Ross and Vasicek models provides the term structure as an output and require as inputs the long run spot rate, the volatility, and the mean reversion parameter. They do not guarantee that an investor could not earn an arbitrage profit trading with current prices. Therefore, we believe that the Heath-Jarrow-Morton model is a superior approach to fitting the term structure.

The HJM model assumes that the uncertainty in the term structure is driven by an arithmetic diffusion process in the forward rates. HJM is consistent with the current term structure and the volatilities of forward rates. For certain volatility structures, closed-form solutions for the prices of bonds and derivatives are available.[15] For arbitrary volatility structures, the model is usually fit using a multinomial tree. The model can accommodate any number of risk factors that drive forward rates, but the more factors used, the more complex and computationally intensive is the model. We limit our analysis to a single factor, to be described and justified later.

III. Empirical Tests Using Actual Data

To compare Eurodollar futures prices with Eurodollar forward prices, we must first determine which futures prices match up with which forward prices. Recall that we have spot prices for 30, 60, 90, …, and 360 days. The Eurodollar time deposit underlying the futures contract has a 90-day maturity. Thus, we can construct forward prices on 90-day Eurodollar time deposits for 30 days ahead, 60 days ahead, 90 days ahead, and so forth to 270 days ahead. We must determine on which days there is a corresponding futures contract expiring in 30, 60, 90, …, and 270 days. We follow this rule strictly and avoid bumping any days forward or backward. For example, if the matching contract requires a weekend quote, it would be tempting to move forward or backward one day to make a match, but we do not do so. The amount of data available provides a reasonable sample size, so we do not distort the results with a few more days to maturity or less in some cases. In all cases our interest calculations are prorated using the ratio "actual/360", which is the custom in the Eurodollar market. It so happens that all contracts we construct have 30, 60, 90, etc. days until expiration. Over

[15]See, for example, Jarrow and Turnbull (2000, Chs. 15-17) who rely on the assumption that the volatilities of forward rates are mathematically related by an exponentially dampening function.

the 1987-2000 period, we are able to obtain 457 matching quotes with maturities of one to nine months, and this would seem to be a sufficient sample size.

The results are summarized in Table I. As stated previously, prices are based on a par value of $1. All of the average differences for the various maturities are negative and statistically significant, as we might expect. The averages range from less than -0.0001 to -0.0011. Note, however, in the last column that anywhere from 17.9 to 30.3 percent of our observations are greater than zero. This result means that the futures price exceeds the forward price and is inconsistent with the Cox-Ingersoll-Ross proposition that the forward price should exceed the futures price because interest rates and Eurodollar futures prices are inversely related. It should be noted that the Cox-Ingersoll-Ross result does not require any restrictions on preferences. It is a strong result, requiring only mild conditions.

Although the magnitudes of the differences are small, the presence of a large number of positive differences is disconcerting. Imagine a simple case of just two observations, one a large positive difference and one a large negative difference of the same magnitude. The average would be zero but the positive difference half of the time would be inconsistent with rational investors, as well as pushing the average toward zero.

These results, however, are contaminated by the fact that the LIBOR time deposit is priced as an add-on instrument, while the Eurodollar futures contract settles as a discount instrument. Perhaps this problem is the source of the irrational positive differences we observe. In an attempt to adjust for this effect, we calculate forward prices as though the underlying LIBOR time deposits were discount instruments. We refer to this price as the "adjusted forward price."

The differences between the futures prices and the adjusted forward prices are shown in Table II. We observe that while the average differences are negative for forward contracts maturing in five to nine months, the average differences are positive for contracts maturing in one to four months. The positive average differences for one- and two-month contracts are significantly different from zero, and the negative average differences for six-, seven-, and nine-month contracts are significantly different from zero. Note also that the percentage of observations greater than zero ranges from 33.9 to almost 95 percent. We can do a rough statistical test for whether the incidence of positive observations is significant. It is not possible to design a statistical test in which the null hypothesis is that *all* observations are negative. We can, however, state that if all observations are negative, then the sample mean should be negative. If the sample mean is not statistically negative, it would cast doubts on these numbers. A simple t-test applied to the means in Table II reveals that the means for expirations of one through five months and also eight months are not significantly negative.[16]

Thus, after attempting to correct for the settlement issue, the problem is even greater. The average differences remain small, but we cannot trust these results. If Eurodollar futures prices exceed forward prices, investors are irrational.

It is tempting to infer that the non-synchroneity of the markets and the data might be the explanation, but there may be more to it than that. We know that because of the differences in the expiration settlement procedures in the two markets, forward and futures contracts whose prices do not converge at expiration are not perfect substitutes, even in a world of constant interest rates.[17] It may be the case that empirical data are simply not capable of providing the answer to this important question. As such, we turn to an alternative approach, one that blends empirical data with numerical estimates of empirical prices under idealized conditions.

IV. Estimating the Forward-Futures Price Differential Using Blended Data

A. Fitting an Arbitrage-Free Tree to Empirical Data

Because of the problems of straightforward estimation of the forward-futures price differential with empirical data, we turn to an alternative approach. We would like to observe prices in an idealized setting, one free from any differences in expiration settlement procedures and where prices are fully synchronized. In a sense, this is what Sundaresan did, using hypothetical input parameters and the Cox-Ingersoll-Ross model. Also, this is what Grinblatt and Jegadeesh attempted using the Cox-Ingersoll-Ross and Vasicek models with parameters estimated from the data. But these input parameters may not be realistic, may not span a sufficiently wide range of possible values, and the Cox-Ingersoll-Ross and Vasicek models do not fit the current term structure (and, thus, admit arbitrage when used in practice). Also, Grinblatt and Jegadeesh do not account for the expiration settlement feature. So results based thereon can provide only a limited, if not biased, picture of the possible differences between futures and forward prices driven by the daily settlement.

The approach used here assumes that the LIBOR data contain the fundamental information necessary to capture the term structure. Using the HJM model, we then fit an arbitrage-free binomial tree to the term structure, using the actual LIBOR data and

[16]The same test applied to the means in Table I reveals that all are significantly negative.
[17]Of course, in a world of constant interest rates, we would have to be referring to forward and futures contracts driven by something other than interest rates.

volatilities estimated from the time series of the LIBOR rates. This procedure can be used to extract futures prices that are theoretically consistent with the LIBOR term structure and interest rate volatilities, conditional upon the HJM model, and the functional form of the model we choose. These prices will, of course, be free of any problems of non-synchroneity.

The forward prices are calculated from the LIBOR data and, as is the case with forward prices, are not specific to the choice of term structure model and the estimates of interest rate volatility. If, however, we calculate futures prices the standard way using the LIBOR quotes, we would still face the problem that the futures contract settles as though the underlying is a discount instrument, while the underlying instrument is actually an add-on instrument. We address this problem by constructing a hypothetical futures contract that settles based on the add-on method that is used in the spot market. We do not claim that this is the actual Eurodollar futures price. It is the price that should exist if the futures contract were designed to settle at expiration exactly as the spot instrument is priced. Thus, this test is a laboratory experiment that blends real world data that have the characteristics of Eurodollar interest rate movements with arbitrage-free prices that would exist in such a market. In so doing, we can more accurately gauge the effect of the different cash flow streams of forward and futures contracts on their prices.[18]

Given the granularity of the data set, we can, without approximation or interpolation, fit an HJM binomial tree with time steps spaced one month apart out to 11 months. The raw data are the LIBOR rates provided by the British Bankers Association for 1, 2, 3, …, 12 months. We denote these as L(0,1), L(0,2), L(0,3), …, L(0,12). From these we can derive the prices of zero coupon bonds. In general for maturity, i, the price of a LIBOR-based zero coupon bond with maturity an integer multiple of 30 at time 0 is

$$B(0,i) = \frac{1}{1 + L(0,i)\left(\dfrac{30i}{360}\right)}.$$

[18]It is tempting to also try this procedure in the opposite direction. Given the LIBOR rates, suppose we could estimate forward prices by constructing an artificial Eurodollar spot instrument, instead of an artificial Eurodollar futures contract, based on the discount method, rather than the add-on method. We could then compute futures prices based on the discount method, as is done in practice, and the comparability problem should go away. The problem with this approach is that the raw information set is the set of spot and forward *prices*, not the rates. Prices are determined in a competitive market. Since prices represent present values of future claims, they reflect time preferences and expected inflation. Interest rates are simply transformations of prices, and multiple interest rate specifications are possible. If the convention in the Eurodollar market changed overnight from add-on to discount, Eurodollar spot prices would not change. Thus, it is imperative that the true Eurodollar spot prices be used. These prices can be obtained only by using the add-on method with the LIBOR rates provided by the BBA.

The values B(0,1), B(0,2), …, B(0,12) comprise the term structure to which the HJM model will be fit. The forward price based on the term structure at time 0 for a one-period bond starting at time i is denoted as B(0,i,i+1) and found as

$$B(0,i,i+1) = \frac{B(0,i+1)}{B(0,i)}.$$

The HJM model is based on continuously compounded rates, so we first require the one period spot rate, f(0,0), which is obtained as[19]

$$f(0,0) = -\ln B(0,1)\left(\frac{365}{30}\right).$$

The forward rates are found as

$$f(0,i) = -\ln B(0,i,i+1)\left(\frac{365}{30}\right).$$

The set of rates f(0,0), f(0,1), …, f(0,11) constitute the rates that will evolve according to a binomial tree fit to be consistent with the stochastic process of an HJM model and the absence of arbitrage opportunities.[20]

The tests in this paper are based on a one-factor HJM model. While the limitations of a one-factor model might raise some questions, empirical results of Litterman and Scheinkman (1991) and Chapman and Pearson (2001) shows that about 88% of the variation in the term structure can be explained by one factor, which is generally thought to be the level of the term structure.[21] Moreover, Dybvig (1997) concludes that a single factor HJM-type model should be acceptable for the short end of the term structure. The present study is exclusively focused on the short end of the term structure, specifically maturities less than one year.

One obvious question is whether the futures prices we obtain are sufficiently representative of actual futures prices. The estimated futures prices are based on marking-to-market on a monthly basis. It is not possible to fit a term structure that is marked to market daily, as in practice, without having a term structure of bonds with expirations of each day for the life of the futures. No such bonds exist. Moreover, as Heath, Jarrow, Morton, and Spindel (1992) note, their model is considered accurate with usually no more than seven time steps. Their results are all the more relevant to this

[19]Note that we use 360 days when working directly with LIBOR rates and 365 days when computing the continuously compounded rate. The use of 360 days with LIBOR rates is the accepted market convention. When converting to continuously compounded rates, any reasonable convention can be used. Typically the process of continuous compounding or discounting uses the formula Future value = Present value*exp(r*time) where time is in years and measured with a full 365 days.

[20]With only 12 spot rates, we can calculate forward rates only up to month 11.

[21]The first factor in term structure models is typically viewed as the level of the term structure. The second is considered to be the slope, and the third is called the curvature.

study, because they are pricing instruments with up to five years to maturity. We are pricing instruments with nine months maturity, using nine time steps. Wall Street firms are known to trade with binomial trees in which the time step is more than one day. Since Eurodollar futures are widely used as a hedging instrument for swaps and interest rate options, it is apparent that a model such as the one estimated here is accepted as a measure of the term structure and the prices of futures contracts.[22]

B. Forward Rate Volatility Inputs

We require volatilities for these forward rates. We denote a given volatility as σ(i,j), which represents the standard deviation at time i of the forward rate applicable to a one-period bond starting at time j. When fitting the initial tree, time i is time 0. Thus, we require an initial set of forward rate volatilities, σ(0,1), σ(0,2), …, σ(0,11). In fitting the tree, these initial volatilities are used to reflect the movement in forward rates for the first time step. Then for the second time step, we would require another set of volatilities, σ(1,2), σ(1,3), …, σ(1,11). For the third time step, we would require the volatilities, σ(2,3), σ(2,4), …, σ(2,11). This would continue until the entire tree is fit. This formulation would create the tree for the first observation in the data set. For the next observation, a new set of LIBORs the next day, we would fit a new tree. This would continue for the approximately 3,500 days of the data set.

We estimate the volatilities of the forward rates using data from the entire period 1987-2000. We compute the one-period-ahead forward rate volatility, the two-period ahead forward rate volatility, and so forth. These are used as our estimates of σ(0,1), σ(0,2), … etc. For a given day's term structure tree, we use the estimate σ(0,i) as the estimate σ(1,i). Thus, the volatility of the forward rate 11 months ahead is the volatility of that same rate 10 months ahead, 9 months ahead, etc. This assumption enables the tree to recombine and avoid the massive number of calculations required in a non-recombining tree. This assumption should not be confused with the assumption of constant volatility. A constant volatility HJM model is equivalent to the Ho-Lee (1986) model and treats all forward rates as having the same volatility.[23] We estimate the volatilities of the different forward rates. We are merely assuming that a given rate for a point in the future has the same volatility as that time point draws nearer. The empirical estimates we present here suggest that this is a reasonable assumption. Consider for example, the forward rate for a one-period bond six periods hence. For the

[22]At the other extreme, the Grinblatt-Jegadeesh results are estimated using the continuous-time Cox-Ingersoll-Ross and Vasicek models and make the assumption of continuous marking-to-market, which is far more frequent than in reality.

[23]Flesaker (1993) finds that the one-factor HJM model with constant volatility, i.e., the Ho-Lee model, performs poorly.

first time step in the tree, this volatility is denoted as σ(0,6). For the second time step, this rate is denoted as σ(1,6). The remaining rates are σ(2,6), σ(3,6), σ(4,6), σ(5,6).[24] Again, we assume that all of these volatilities are the same. This might not be a reasonable assumption if we find empirically that the volatilities of the forward rates are not relatively close.

Table III provides the estimates of the statistical parameters of the one-period continuously compounded forward rates for various periods ahead. We see that the volatilities of forward rates for various periods ahead are relatively stable, ranging from 1.51% to 1.68%. Note also the small variation in means, medians, maxima, and minima. These results suggest considerable stability in the statistical parameters of short-term forward rates. Perhaps this result should not be surprising. We are fitting a binomial tree to only the first 12 months of the term structure. If we were working with a much longer term structure, there might be more concern.

Thus, in fitting only the short end of the term structure, it seems reasonable that we conclude that while the volatilities of forward rates differ, the volatility of a given forward rate is reasonably stable across time. For the first round of tests, we shall use the volatility estimates in Table III for the appropriate rates in fitting our term structures. In a second round of tests, we shall vary the volatility over the years in the sample.

C. Estimation of the Drift

As is well-known in the term structure literature, the arbitrage-free condition, also known as the Local Expectations Hypothesis, is equivalent to a specific drift formula. For the HJM model, the drift formula is determined completely by the volatilities of the forward rates. To obtain the drift, we use the technique of Grant and Vora (1999), which is briefly outlined here.

Consider a given forward rate f(t,T), which is the forward rate observed at time t for a one-period bond to begin at time T. In a binomial tree, the change in the forward rate is

$$\Delta f(t,T) = \alpha(t,T)h + \sigma(t,T)\Delta W(t,T)\sqrt{h},$$

where h is the length of the time interval, α(t,T) is the drift or expected change in the forward rate, σ(t,T) is the volatility of the forward rate, and ΔW(t,T) is a discrete time random walk that takes on a value of –1 or +1 with equal probability. Without loss of generality, h can be standardized to a value of 1. Consistent with HJM and the

[24]There is no σ(6,6) because the one-period rate at time 6 is the spot rate, which has no volatility. It is just the rate on a zero coupon bond expiring in one period.

requirement of no arbitrage, Grant and Vora show that the drift term, α(t,T), is constrained by the volatility and is given by the following formula:

$$\alpha(t,T) = \sigma(t,T)\sum_{j=t+1}^{T}\sigma(t,j) - \frac{1}{2}\sigma(t,j)^2.$$

They then show that this term can be obtained from the covariance matrix of forward rate volatilities as one-half the sum of the T^{th} row and column. Alternatively, it can be calculated directly from the formula above.

D. Fitting the Tree and Estimating the Futures Price

For a set of forward rates of 1, 2, …, 11 periods ahead, we can obtain a tree of up to 11 periods. We need estimates of the prices of futures contracts on a three-month Eurodollar time deposit. Thus, the data available allow us to obtain these estimates for up to nine months to expiration. In other words, at time 9, we would have the current one-period spot rate, the one-period ahead one-period forward rate, and the two-period ahead one-period forward rate. With each period being one month, this information would be sufficient to price a three-month spot Eurodollar time deposit. The rate on this instrument would be the rate from which the futures contract expiring in nine months is priced at expiration. We can, thus, price futures contracts with 1, 2, …, 9 months to expiration where the underlying is a three-month Eurodollar.

To illustrate, for a given day, starting in January 1987, we fit a one-factor HJM model to the tree of LIBOR rates extending out for nine months. We then calculate the price at time 0 of a futures contract expiring in 1, 2, …, 9 months. The procedure is obtained by taking the spot price at expiration of a three-month Eurodollar. Suppose at an expiration point denoted as time T, the one-month continuously compounded spot rate is 6%, the one-month continuously compounded forward rate one-month ahead is 6.25%, and the one-month continuously compounded forward rate two months ahead is 6.5%. Thus, the price of a three-month Eurodollar time deposit paying $1 at maturity is

$$\exp[-.06(30/365)]\exp[-.0625(30/365)]\exp[-.065(30/365)] = 0.9847.$$

This price implies a spot 90-day LIBOR of

$$L(T,T+90) = \left(\left(\frac{1}{0.9847}\right) - 1\right)\left(\frac{360}{90}\right) = .0622.$$

Assuming the standard settlement procedure, the price of an expiring futures contract would then be

$$F(T,T) = 1 - .0622\left(\frac{90}{360}\right) = 0.9845.$$

Observe the difference between the forward price (0.9847) and the futures price (0.9845). Since this is an expiration price, this difference is unaffected by the daily settlement process and is entirely attributable to the settlement procedure used to determine the futures price at expiration. As we noted in Section IB, this differential would be 1 to 10 basis points for a reasonable range of LIBORs.

To obtain the futures price prior to expiration, we start at expiration where the futures price is determined for all possible binomial outcomes. Then, stepping back in the tree, the futures price one period prior is the martingale probability-weighted average of the next two possible futures prices. We then repeat the procedure, stepping back through all outcomes and time steps until reaching time 0, at which point we have the futures price.

As noted above, the forward prices are easy to obtain directly from the spot prices. Hence, they are not influenced by the chosen term structure model or the volatility estimates. As noted, however, the standard forward prices are based on the assumption that the Eurodollar time deposit is an add-on instrument. If we compare them to the futures prices, wherein the underlying is treated as though it were a discount instrument, we are subject to the expiration settlement effect. We shall, therefore, have to address this concern.

V. Empirical Results of the Forward-Futures Price Differential Using an Arbitrage-Free Binomial Tree

In this section we provide the results for estimates of the differences between forward and futures prices using the HJM model fit to daily term structure data over 1987-2000 under two conditions. In the first set of results we estimate the futures price under the assumption that the contract settles in the manner it does in practice, that is, as though the underlying is a discount instrument. We remove that bias in the second set of results by forcing the contract to settle at expiration in the add-on manner. Although the first set of results is a better approximation of reality, the second set of results more accurately captures the effect of the daily settlement that is the primary distinction between the cash flow streams of futures contracts and forward contracts.

In all of the results that follow, we conducted t-tests for the significance of the mean differences. These differences are overwhelmingly significant, with most of the t-statistics greater than 10 and some more than 10,000. But these statistics are not reliable, because we do not know the sampling properties of these estimates. Our testing procedure is not equivalent to drawing a series of independent random samples. Therefore, the inferences that follow will be drawn less formally.

As we examine these differences, we should keep in mind that (per discussions with the CME), a typical bid-ask spread on the futures is about 0.5 basis points.

A. Differences in Forward and Estimated Futures Prices under the Assumption that Futures Contracts Settle Using the Discount Method

Table IV shows that the average difference between forward and futures prices is 2.6 to 7 basis points with a standard deviation of about 1.2 basis points. The smallest difference is in futures contracts expiring in one month. The largest is in contracts expiring in nine months. Notice that the range is, however, relatively large. For contracts expiring in one month the maximum is about 10 times the minimum. For contracts with later expirations, however, the maximum is about twice the minimum. The differences across all expirations range from 0.7 to about 12 basis points, which exceed the bid-ask spread. Nonetheless, these results do show smaller differences than reported by Meulbroek, are similar to those reported by Grinblatt and Jegadeesh, and slightly smaller than those we reported in Section III, all using empirical data. In addition, these results are free of any non-synchroneity problems or biases in empirical data due to market imperfections. Most importantly, none of these results are positive, as is typically observed in empirical data and would be completely inconsistent with rational investors.

It is worthwhile to note, however, that even though the mean difference is smallest for contracts expiring in one month, the standard deviation is the largest and the range is considerably larger than for contracts with later expirations. Ordinarily, pricing a futures contract expiring in the first time step of a binomial tree would lead to a futures price exactly equal to the forward price, because there is no interim daily settlement. Yet in this case, not only are the two not equal, but the one-month contract has the largest standard deviation and a maximum difference ten times its minimum. More stable results are obtained for contracts maturing at later time steps. In fact, the most reliable results are obtained for contracts maturing the latest. These subtle findings suggest that something is awry. Indeed, the expiration settlement feature of Eurodollar futures contracts is the culprit.

B. Differences in Forward and Estimated Futures Prices under the Assumption that Futures Contracts Settle Using the Add-on Method

To remove the bias from the fact that Eurodollar futures settle by the discount method, we construct a hypothetical futures contract that settles by the add-on method and is priced off of the term structure, the evolution of which is modeled by the Heath-Jarrow-Morton model. The results are presented in Table V.

The first point to notice is the results for the futures contract expiring in one month, or equivalently, one binomial time step. As noted in Section V.A., a futures contract with a single time period would be equivalent to a forward contract, and, therefore, should have the same price. We do observe a very small difference, but it is not measurable until the 6th digit and amounts to about 0.08 basis points. The fact that it is not precisely zero can be easily explained. The Grant-Vora technique to estimate the drift is based on the assumption that a binomial tree converges to a continuous-time normal distribution in the limit. Clearly, this convergence cannot occur with one time step. Nonetheless, the difference is extremely small and for all intents and purposes is zero. The standard deviation is undetectable through seven digits, and there is barely a difference between the extremely small maximum and minimum. These results for a one-month futures contract are something of a litmus test for the reliability of this estimation procedure, and it appears as if it passes the test.

For the remaining contracts, the mean differences range from –0.26 basis points to –4.1 basis points, with the magnitudes of the differences increasing with contract maturity. The mean differences in Table IV, however, are from about 2 to over 33 times the mean differences in Table V, meaning that the effect of the futures contract settling as a discount instrument has a significant impact on the estimated forward-futures differential. Some of these values do exceed the bid-ask spread and some are less.

The standard deviations in Table IV are from over 80 to over 4,000 times the corresponding ones in Table V, where the largest standard deviation is but 0.015 basis points. We remarked on the ranges in Table IV. In Table V, the differences between maximum and minimum values are extremely small for all maturities. These results are remarkably stable, especially considering they are estimated from over 3,500 term structures over 14 years.[25] Thus, while the magnitudes of the differences between futures and forward prices attributed to the daily settlement are smaller once the expiration settlement feature is eliminated, the reliability of the estimated differences is also considerably greater.

C. Improved Estimates Using Differential Volatilities Across Time

As noted earlier, we used volatility estimates that were based on an overall volatility for the sample period. We obtained a volatility for the one-month ahead forward rate, the two-month ahead forward rate, on up to the nine-month ahead forward rate. The forward rate volatilities do vary considerably from year to year, as indicated

[25] It is tempting to suggest that the large sample size leads to stability of these estimates, but the same term structures are used to estimate the results in Table IV, which are much less reliable.

in Figure 2. We see that forward rate volatilities were relatively high in certain years and much lower in others. Of course the differential volatility across forward maturities is taken care of by our using different volatilities for different forward rate maturities. But the differences in volatilities across time have not been considered.

Accordingly, the tests are rerun in two ways. We run the tests for the overall period (1987-2000) but use a different set of forward rate volatilities each year. These volatilities are estimated from the data for that specific year. For example, when we run the tests using data from a given year, we use forward rate volatilities estimated using data from that same year. Then we also run the tests separately for each year in the overall period, using, of course, the appropriate forward rate volatilities for the respective years. These tests are done using the hypothetical futures contract, which settles based on the add-on method. The results are presented in Table VI. Due to the large amount of output, we show only the average differences.

The column labeled "Overall" gives the results for the full time period, using the appropriate volatility for each year within that period. The largest average is about 0.7 basis points. The mean differences in Table V, obtained without varying the volatility by year, are anywhere from 5.8 to 9.6 times the mean differences for the overall period in Table VI, where the volatility is varied appropriately by year.[26] The mean differences always increase as the maturity increases, a result we observed in both previous tables.

Another interesting finding is that if the average differences for each of the 14 years are sorted in ascending order, the ordering corresponds identically to that if the average of the estimated forward rate volatilities for each of the 14 years is also sorted in ascending order. In other words, without fail, if interest rates are more volatile in a particular year, the estimated differences between forward and futures prices is greater. This point was also observed by Grinblatt and Jegadeesh in their data.

The remaining columns in Table VI show the results year-by-year with the appropriate volatility used for the respective year. The largest difference over the entire table is 1.9 basis points. In 10 of the 14 years, the largest difference is less than one basis point. Seven of the 14 largest differences are less than the bid-ask spread.

[26]Although not shown, the results when the volatilities are varied by year (Table VI) have higher standard deviations than in the case where the volatilities are not varied by year (Table V). This result should not surprising, however, since the altering of volatility from year to year should result in less stability in the estimates. The volatilities are still considerably smaller than those reported in Table IV where the expiration settlement feature corrupts the results. Tests in which the futures settles by the add-on method but the volatility varies by year resulted in standard deviations that are over 2.8 to 165 times those obtained when the expiration settlement feature is not a factor and the volatility is varied year by year. So, when the volatility is varied by year, the standard deviations will be higher than when a single volatility is used for each forward rate over the full time period, regardless of how the expiration settlement feature is accounted for.

D. The Effect of Estimating Volatilities from Historical Data

Although the Heath-Jarrow-Morton model assumes a deterministic volatility structure, in most financial models the parameters must be estimated, which nearly always requires the use of historical data. In this section we examine the effect of estimating the volatilities from historical data.

First, let us recall that the results of this study are based on one source of input information, the set of spot prices, derived from the term structure of spot rates, of Eurodollar time deposits. From this information, we estimated the forward rates and from these forward rates, we estimated their volatilities. As shown in the previous section, the use of more contemporaneous estimates of volatilities leads to much smaller estimates of the difference between forward and futures prices. This volatility information would not, however, be known to traders at the time at which they are computing arbitrage-free prices. Traders would have had to forecast the volatilities.

There are an infinite number of possible forecasting models. The objective of this paper is not to address the myriad of complex issues associated with interest rate volatility forecasting.[27] Thus, we examine the effect of a very simple forecasting model. We take the series of daily forward rates for one year, estimate the volatility of each forward rate, use that estimate to fit the current term structure, and conduct the tests using the following year's data. This procedure is updated each year. Given the data set that covers 1987-2000, the tests span the period 1988-2000. We conduct these tests using the assumption that the futures contract is constructed as an add-on instrument.

A representative sample of the results is shown in Table VII where we show the findings for the entire time period and for the years 1988, 1992, 1996 and 2000 for futures contracts expiring in one month, three months, six months, and nine months. The omitted results follow the same patterns. For each combination of contract maturity and time period, we show two numbers. The top number is the difference between the futures and forward prices using the volatility forecasted for the given time period. The number below and in parentheses is the number, taken from Table VI, representing the average difference between the futures and forward prices where the volatility is estimated contemporaneously. For example, for a six-month contract in 1992, the average difference obtained when fitting 1992 term structure data using volatilities estimated with 1991 data is –0.476 basis points. The average difference obtained when fitting 1991 term structure data using volatilities estimated with 1991 data is –0.473 basis points.

[27]See, for example, Brenner, Harjes, and Kroner (1996) and Chan, Karolyi, Longstaff, and Sanders (1992).

First observe the results for the overall time period. When forecasting the volatilities, the differences are still negative but slightly more negative. Nonetheless, the differences are on a very similar order of magnitude to those obtained with contemporaneous volatility estimates. Thus, even with a very crude forecasting model, we obtain virtually the same results as if we had perfect foresight of contemporaneous volatility.[28] For each individual year, observe that the average differences are quite close to the average differences of the previous year when contemporaneous volatility estimates are used. These findings are consistent throughout Table VII and also in the results not reported in the table and imply that the volatility of forward rates and not the level of the term structure is the primary driver in determining the difference between futures and forward prices. Yet, even when volatility must be forecasted and only an extremely simple model is used, the results are almost indistinguishable from those that use contemporaneous volatility, which would require perfect foresight.

VI. Conclusions

There has been very little empirical research on the differential between forward and futures prices and even less for the most active forward and futures market, the Eurodollar/LIBOR market. But what published empirical tests tell us is that the differential between Eurodollar forward and futures prices is somewhere from 2 to 48 basis points. A closer look at these results also suggests that a large number of these differences are positive, meaning that futures prices exceed forward prices, a result that implies irrational investors and suggests that the Cox-Ingersoll-Ross proposition is not well supported. Moreover, the differences are quite large and exceed the bid-ask spread.

If we generalize this finding to all forward and futures markets, we must be careful. The CIR proposition requires that the futures price converges to the price of the underlying spot asset at expiration. The expiration settlement feature of Eurodollar futures contracts, however, is based on the discount method, while the Eurodollar time deposit underlying the futures is based on the add-on method. This discrepancy in settlement methods of spot and futures is unique to the Eurodollar contract. We show that this problem creates a large differential between futures and spot prices at expiration that also affects the pricing prior to expiration. After adjusting for this

[28] The only other factor that could have an impact on these comparisons is the fact that the tests using forecasted volatilities cover the 1988-2000 period and the tests using the contemporaneous volatilities cover the additional year of 1987. The tests using contemporaneous volatilities for the overall period but excluding 1987 resulted in average differences for one, three, six, and nine months of –0.007, -0.062, -0.264, and –0.656 basis points. Thus, removing the 1987 results with contemporaneous volatilities leads to similar but slightly smaller differences. Hence, these results with forecasted volatilities are slightly more negative when 1987 is removed from the benchmark, but they are still on a very similar order of magnitude.

problem, the difference between forward and futures prices is extremely small. Seven of the 14 largest annual average differences using contemporaneous estimates of the volatility are less than the bid-ask spread, and most of the remaining average differences do not exceed the typical bid-ask spread by much. The overall average difference of about 0.7 basis points is only slightly higher than the bid-ask spread. Moreover, we show that volatility can be forecasted sufficiently well so that these results are supported with only a simple forecasting model.

In short, if we conclude that after adjusting for the expiration settlement effect, we find a rough average price differential of one basis point (conservatively adjusting upward), this finding is notably less than the 2 to 48 basis point range found in other studies.

We have also found that the differences between futures and forward prices in the Eurodollar market are consistently negative, as they should be, and extremely close to zero. It seems reasonable to infer that in markets in which the linkage between futures prices and interest rates is weaker and less direct, the differences attributable to the daily settlement would be even closer to zero. Of course, other factors such as credit risk (see Jarrow and Turnbull (1997) and Murawski (2003) for analyses) can account for a difference between Eurodollar forward and futures prices, and transaction costs, taxes, and other market imperfections can have an impact. But the differential cash flow streams arising from the daily settlement appears to have only a modest impact on pricing in Eurodollar markets.

Table I
Statistical Properties of the Difference Between Eurodollar Futures Prices and Forward Prices

Spot LIBOR rates are obtained from the British Bankers Association web site (www.bba.org), which consists of rates for each business day for Eurodollar time deposits of 1 to 12 month maturities, covering the period 1987-2000. Forward prices are then inferred from the implied forward rates. Futures prices are for the 90-day Eurodollar contract of the Chicago Mercantile Exchange, which are obtained from the Institute for Financial Markets, and cover the same period. The expiration dates of the futures contracts are matched with the expiration dates of forward contracts and the difference, calculated as "futures price – forward price", is obtained. Means, standard deviations, maximums, and minimums are in basis points.

Expiration (months)	Number of observations	Mean	Standard Deviation	t-statistic	Maximum	Minimum	% > 0
1	39	-0.1148	0.1861	-3.85	5.2077	-4.8757	17.9
2	92	-2.2850	3.4778	-6.30	9.8398	-1.0844	21.7
3	56	-3.6640	5.5640	-4.93	7.7707	-19.1365	21.4
4	33	-2.5810	7.4023	-2.00	12.7200	-1.4312	30.3
5	76	-4.7770	8.1537	-5.11	15.9443	-26.3742	28.9
6	44	-8.3230	10.4551	-5.28	14.4599	-31.3581	18.2
7	46	-8.4700	10.7234	-5.36	11.6936	-30.3660	19.6
8	15	-4.2830	0.9199	-1.80	18.1981	-18.4995	26.7
9	56	-11.2650	13.3802	-6.30	15.8212	-44.6081	19.6

Table II
Statistical Properties of the Difference Between Eurodollar Futures Prices and Adjusted Forward Prices

Spot LIBOR rates are obtained from the British Bankers Association web site (www.bba.org), which consists of rates for each business day for Eurodollar time deposits of 1 to 12 month maturities, covering the period 1987-2000. Futures prices are for the 90-day Eurodollar contract of the Chicago Mercantile Exchange, which are are obtained from the Institute for Financial Markets and cover the same period. The expiration dates of the futures contracts are matched with the expiration dates of forward contracts and the difference, calculated as "futures price – forward price", is obtained. The adjusted forward price is based on the assumption that Eurodollar time deposits are priced as discount instruments, whereby the interest is deducted from the face value in advance, in contrast to the standard procedure in which the interest is added on to the principal. Means, standard deviations, maximums, and minimums are in basis points.

Expiration (months)	Number of observations	Mean	Standard Deviation	t-statistic	Maximum	Minimum	% > 0
1	39	2.4143	1.7615	8.56	8.9982	-0.8701	94.9
2	92	1.5831	3.4554	4.39	13.6408	-6.9536	70.7
3	56	0.4568	5.8859	0.58	16.1452	-14.6018	44.6
4	33	1.0291	7.2887	0.81	15.6575	-10.3969	54.5
5	76	-0.7620	8.4474	-0.79	18.8027	-22.0896	47.4
6	44	-4.0060	11.1674	-2.38	20.9214	-26.9812	36.4
7	46	-4.3380	11.2917	-2.61	17.6678	-27.5036	37.0
8	15	-0.8320	9.1025	-0.35	21.0565	-14.6519	40.0
9	56	-7.0740	14.0380	-3.77	21.1328	-40.2312	33.9

Table III
Statistical Properties of Daily LIBOR Continuously Compounded Forward Rates, 1987-2000.

Spot LIBOR rates are obtained from the British Bankers Association web site (www.bba.org), which consists of rates for each business day for Eurodollar time deposits of 1 to 12 month maturities, covering the period 1987-2000. Forward rates are computed from spot rates and converted to continuous compounding. The number of observations is slightly more than 3,500.

Forward rate (f(0,j) where j is expiration)	Mean	Standard Deviation	Median	Maximum	Minimum
f(0,1)	6.14%	1.68%	5.87%	10.63%	3.16%
f(0,2)	6.19%	1.66%	5.92%	11.14%	3.15%
f(0,3)	6.20%	1.63%	5.92%	10.94%	3.14%
f(0,4)	6.25%	1.61%	5.96%	11.27%	3.13%
f(0,5)	6.25%	1.60%	5.98%	11.22%	3.12%
f(0,6)	6.32%	1.57%	6.07%	11.19%	3.30%
f(0,7)	6.34%	1.56%	6.08%	10.99%	3.23%
f(0,8)	6.39%	1.56%	6.10%	11.24%	3.16%
f(0,9)	6.44%	1.51%	6.24%	11.02%	3.33%
f(0,10)	6.47%	1.50%	6.24%	11.02%	3.32%
f(0,11)	6.50%	1.51%	6.26%	11.03%	3.20%

Table IV
Statistical Properties of the Difference Between the Estimated Futures Price and the Forward Price, LIBOR, 1987-2000

Spot LIBOR rates are obtained from the British Bankers Association web site (www.bba.org), which consists of rates for each business day for Eurodollar time deposits of 1 to 12 month maturities, covering the period 1987-2000. Forward prices are then inferred from the implied forward rates. A one-factor Heath-Jarrow-Morton is fit to the term structure of LIBOR spot rates using a binomial tree. Volatilities are estimated from forward rate data over the entire time period. The tree is fit using nine time steps, each representing one month. The futures prices is obtained by starting at the expiration with the three-month spot LIBOR, applying the equivalent martingale principle and working backwards through the tree to time 0. The number of observations is slightly more than 3,500. Means, standard deviations, maximums, and minimums are in basis points.

Expiration (months)	Mean	Standard Deviation	Maximum	Minimum
1	-2.5800	1.2992	-7.1965	-0.7112
2	-2.8033	1.2881	-7.6274	-0.9092
3	-3.1211	1.2742	-8.0804	-1.1934
4	-3.6097	1.2767	-8.6127	-1.6713
5	-4.1009	1.2613	-8.9856	-2.1685
6	-4.6981	1.2564	-9.4315	-2.7430
7	-5.3865	1.2457	-10.1260	-3.4130
8	-6.1536	1.2358	-10.8960	-4.1677
9	-7.0040	1.2316	-11.7500	-4.9897

Table V
Statistical Properties of the Difference Between the Estimated Futures Price Based on Add-on Interest and the Forward Price, LIBOR, 1987-2000

Spot LIBOR rates are obtained from the British Bankers Association web site (www.bba.org), which consists of rates for each business day for Eurodollar time deposits of 1 to 12 month maturities, covering the period 1987-2000. Forward prices are then inferred from the implied forward rates. A one-factor Heath-Jarrow-Morton is fit to the term structure of LIBOR spot rates using a binomial tree. Volatilities are estimated from forward rate data over the entire time period. The tree is fit using nine time steps, each representing one month. The futures prices is obtained by starting at the expiration with the three-month spot LIBOR, applying the equivalent martingale principle and working backwards through the tree to time 0. The futures price settles at expiration based on the assumption of add-on interest, thereby removing the expiration settlement effect. The number of observations is slightly more than 3,500. Means, standard deviations, maximums, and minimums are in basis points.

Expiration (months)	Mean	Standard Deviation	Maximum	Minimum
1	-0.0766	0.0000	-0.0772	-0.0758
2	-0.2602	0.0010	-0.2615	-0.2572
3	-0.5446	0.0022	-0.5487	-0.5381
4	-0.9810	0.0072	-1.0032	-0.9667
5	-1.4261	0.0000	-1.4283	-1.4244
6	-1.9621	0.0075	-1.9770	-1.9399
7	-2.5927	0.0098	-2.6124	-2.5635
8	-3.2976	0.0123	-3.3225	-3.2607
9	-4.0851	0.0150	-4.1162	-4.0395

Table VI
Average Basis Point Differences Between Futures and Forward Prices Using Different Volatilities Overall and by Year

Spot LIBOR rates are obtained from the British Bankers Association web site (www.bba.org), which consists of rates for each business day for Eurodollar time deposits of 1 to 12 month maturities, covering the period 1987-2000. Forward prices are then inferred from the implied forward rates. A one-factor Heath-Jarrow-Morton is fit to the term structure of LIBOR spot rates using a binomial tree. Volatilities are estimated from forward rate data separately for each year. The tree is fit using nine time steps, each representing one month. The futures prices is obtained by starting at the expiration with the three-month spot LIBOR, assuming that the futures price is based on the add-on method of interest, applying the equivalent martingale principle to work backwards through the tree to time 0. Thus, the futures price is arbitrage-free and free of the expiration settlement effect. The results labeled "Overall" cover the entire time period of 1987-2000 but calculations for observations of a given year use forward rate volatilities that are estimated using data for that year. The number of observations is slightly more than 3,500.

Expiration (months)	Overall	1987	1988	1989	1990	1991	1992	1993
1	-0.0081	-0.0167	-0.0196	-0.0137	-0.0022	-0.0108	-0.0039	-0.0008
2	-0.0295	-0.0577	-0.0652	-0.0547	-0.0095	-0.0437	-0.0160	-0.0033
3	-0.0663	-0.1204	-0.1336	-0.1315	-0.0237	-0.1049	-0.0365	-0.0082
4	-0.1684	-0.2799	-0.2713	-0.3139	-0.1096	-0.2492	-0.1051	-0.0426
5	-0.2169	-0.4004	-0.3212	-0.4302	-0.1124	-0.3451	-0.1380	-0.0383
6	-0.2852	-0.5549	-0.3666	-0.5928	-0.1325	-0.4733	-0.1954	-0.0374
7	-0.3995	-0.7703	-0.4670	-0.7994	-0.1954	-0.6655	-0.3047	-0.0581
8	-0.5405	-1.0308	-0.5756	-1.0948	-0.2751	-0.8518	-0.4386	-0.0924
9	-0.7053	-1.3434	-0.7166	-1.4218	-0.3778	-1.1425	-0.5791	-0.1347
Largest (<0)	-0.7053	-1.3434	-0.7166	-1.4218	-0.3778	-1.1425	-0.5791	-0.1347
Smallest (<0)	-0.0000	-0.0167	-0.0196	-0.0142	-0.0022	-0.0108	-0.0039	-0.0008

(continued)

Table VI (continued)

Expiration (months)	1994	1995	1996	1997	1998	1999	2000
1	-0.0254	-0.0028	-0.0009	-0.0004	-0.0018	-0.0107	-0.0023
2	-0.0895	-0.0141	-0.0042	-0.0016	-0.0079	-0.0372	-0.0081
3	-0.2016	-0.0394	-0.0123	-0.0039	-0.0213	-0.0707	-0.0189
4	-0.4113	-0.1322	-0.0698	-0.0513	-0.0836	-0.1491	-0.0849
5	-0.6137	-0.1826	-0.0742	-0.0366	-0.0939	-0.1615	-0.0836
6	-0.8534	-0.2770	-0.0901	-0.0245	-0.1149	-0.1808	-0.0916
7	-1.1626	-0.4392	-0.1418	-0.0375	-0.1689	-0.2389	-0.1342
8	-1.5325	-0.6622	-0.2158	-0.0546	-0.2335	-0.3082	-0.1900
9	-1.8973	-0.9022	-0.3049	-0.0718	-0.3122	-0.3956	-0.2612
Largest (<0)	-1.8973	-0.9022	-0.3049	-0.0718	-0.3122	-0.3956	-0.2612
Smallest (<0)	-0.0254	-0.0028	-0.0009	-0.0004	-0.0018	-0.0107	-0.0023

Table VII
Average Basis Point Differences Between Futures and Adjusted Forward Prices Using Different Volatilities Overall and for Selected Years

Spot LIBOR rates are obtained from the British Bankers Association web site (www.bba.org), which consists of rates for each business day for Eurodollar time deposits of 1 to 12 month maturities, covering the period 1987-2000. Forward prices are then inferred from the implied forward rates. A one-factor Heath-Jarrow-Morton is fit to the term structure of LIBOR spot rates using a binomial tree. Volatilities are estimated from forward rate data separately for each year. The data from a given year are used to estimate the volatilities of each forward rate, which are then used as volatilities for the following year. The tree is fit using nine time steps, each representing one month. The futures prices is obtained by starting at the expiration with the three-month spot LIBOR, assuming that the futures price is based on the add-on method of interest, applying the equivalent martingale principle to work backwards through the tree to time 0. Thus, the futures price is arbitrage-free and free of the expiration settlement effect. The results labeled "Overall" cover the entire time period of 1988-2000 but calculations for observations of a given year use forward rate volatilities estimated using data for the previous year. The overall number of observations is about 3,300. The number in the top of the cell is the average difference in the futures and forward price. The number in the bottom of the cell is the average difference in the futures and forward price from the previous year (see Table VI), where the volatility is estimated contemporaneously.

Expiration (months)	Overall	1988	1992	1996	2000
1	-0.0085 (-0.0081)	-0.0167 (-0.0167)	-0.0108 (-0.0108)	-0.0028 (-0.0028)	-0.0107 (-0.0107)
3	-0.0698 (-0.0663)	-0.1202 (-0.1204)	-0.1054 (-0.1049)	-0.0394 (-0.0394)	-0.0705 (-0.0767
6	-0.3000 (-0.2852)	-0.5540 (-0.5549)	-0.4757 (-0.4733)	-0.2772 (-0.2770)	-0.1804 (-0.1808)
9	-0.7399 (-0.7053)	-1.3411 (-1.3434)	-1.1476 (-1.1425)	-0.9030 (-0.9022)	-0.3955 (0.3956)

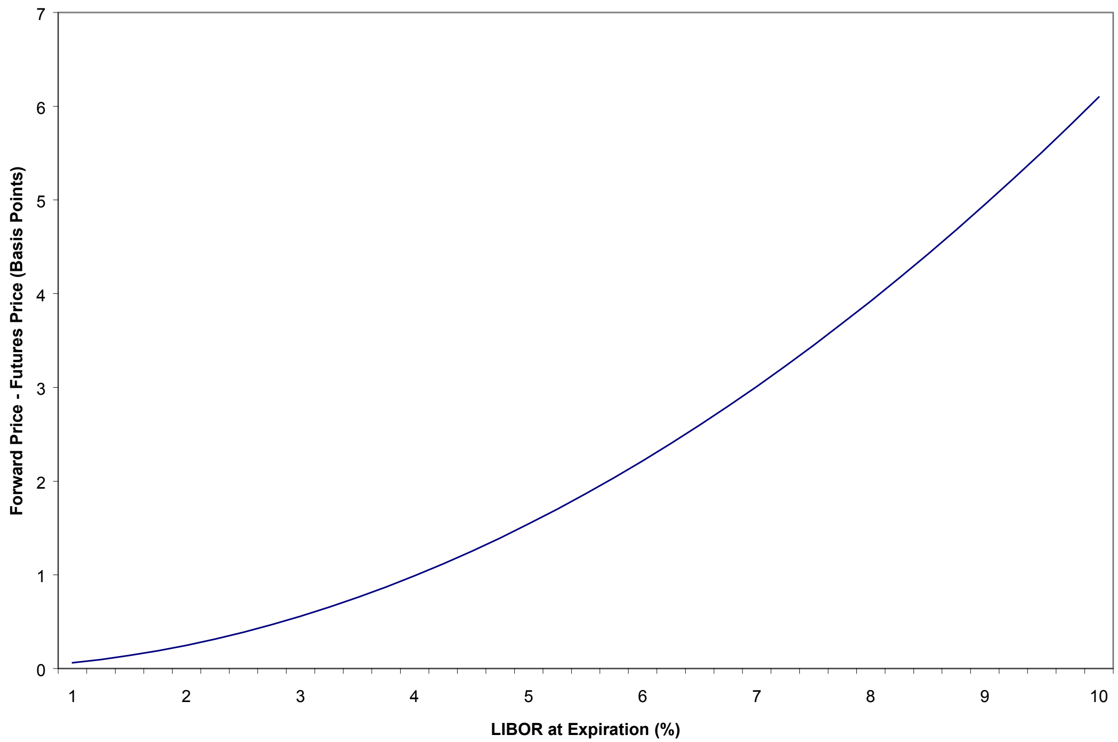

Figure 1. The Difference Between the Forward and Futures Price per $1 Par at Expiration for a Range of LIBOR

The forward price is computed as $1/(1 + \text{LIBOR}(90/360))$, and the futures price is computed as $1 - \text{LIBOR}(90/360)$.

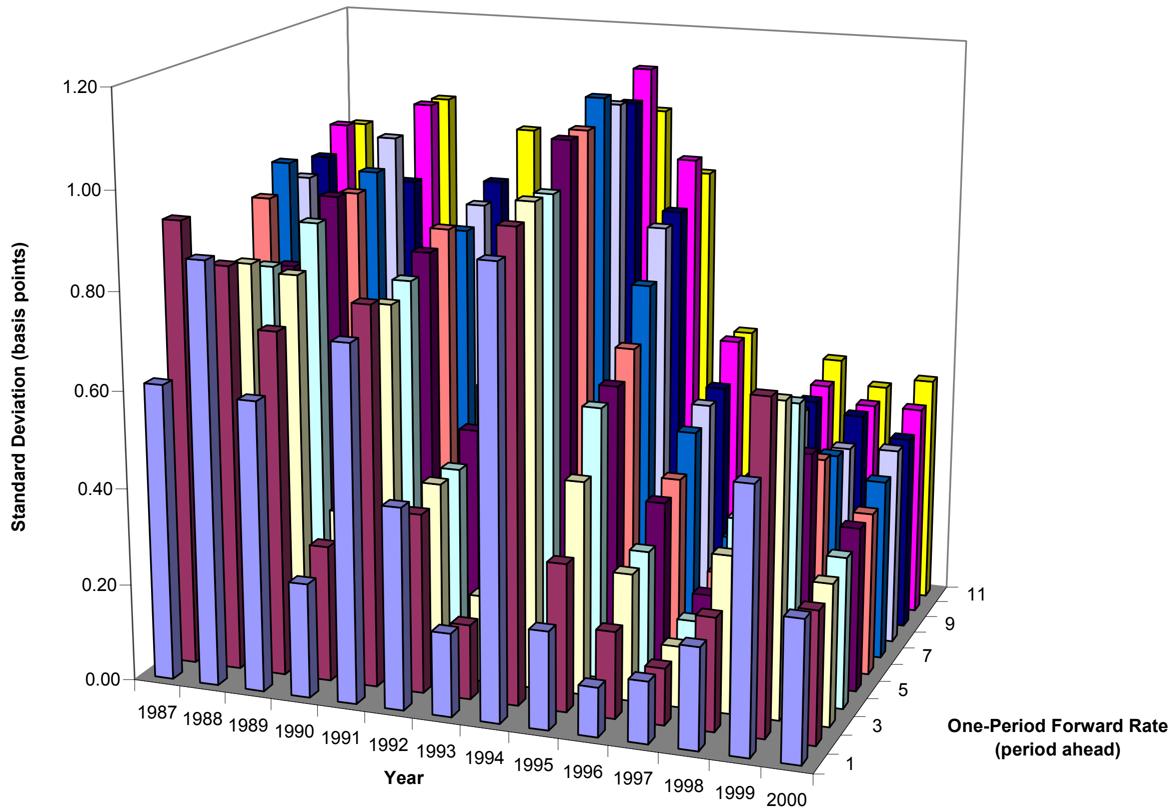

Figure 2. Year-by-Year Volatility Estimates. Spot LIBOR rates are obtained from the British Bankers Association web site (www.bba.org.uk), which consists of rates for each business day for Eurodollar time deposits of 1 to 12 month maturities, covering the period 1987-2000. Forward rates are computed from spot rates and converted to continuous compounding. The number of observations is slightly more than 3,500.

Appendix A

Here we derive the Cox-Ingersoll-Ross proposition for Eurodollar forward and futures contracts. Let today be time 0, and the futures and forward contract expirations be time T. Let F(0,T) be the price fixed at time 0 of a forward contract expiring at time T. The price at time t of a futures at time T is $f_t(T)$. The underlying stochastic variable is m-day LIBOR, denoted as $L_T(m)$ at time T. Let $B_t(t+1)$ be the price at time t of a one-period Eurodollar time deposit maturing, which is given as $1/(1 + L_t(t+1)(1/360))$. Following CIR's Proposition 6, we go short a forward contract and long $B_t(t+1)$ futures contracts. Each period, j = 1, …, T-1, we liquidate the futures, invest the gains or losses in a one-period Eurodollar time deposit, and revise the number of contracts to the new value of the one-period Eurodollar spot price. At expiration, the forward contract pays off $-(B_T(T+m) - F(0,T))$. The value of the overall strategy at T will be

$$V_T = -(B_T(T+m) - F(0,T)) + \sum_{t=0}^{T-1}\left(\frac{1}{1+L_t(t+1)(1/360)}\right)[f_{t+1}(T) - f_t(T)](1 + L_{t+1}(t+2)(1/360).$$

This expression can be re-written as

$$V_T = -(B_T(T+m) - F(0,T)) + \sum_{j=0}^{T-1}[f_{t+1}(T) - f_t(T)]$$
$$+ \sum_{j=0}^{T-1}[f_{t+1}(T) - f_t(T)]\left(\frac{1+L_{t+1}(t+2)(1/360)}{1+L_t(t+1)(1/360)} - 1\right).$$

The first summation term on the right-hand side reduces to $f_T(T) - f_0(T)$, leaving the entire expression as

$$V_T = -B_T(T+m) + f_T(T) + F(0,T) - f_0(T)$$
$$+ \sum_{j=0}^{T-1}[f_{t+1}(T) - f_t(T)]\left(\frac{1+L_{t+1}(t+2)}{1+L_t(t+1)} - 1\right).$$

The first two terms on the right-hand side are $-B_T(T+m)$, the Eurodollar spot price at expiration, and $f_T(T)$, the futures price at expiration. CIR require that these terms offset. If they do, the difference between forward and futures price would be driven by the summation term on the right-hand side, which could be modeled as the covariance between LIBOR and the Eurodollar futures prices. In that case, with the covariance clearly negative, $V_T = 0$ only if $F(0,T) > f_0(T)$, i.e., the forward price exceeds the futures price. For Eurodollar transactions, however, the first two terms are

$$B_T(T+m) = \frac{1}{1+L_T(m)(m/360)} \neq f_T(T) = 1 - L_T(m)(m/360).$$

Thus, the expression does not simplify sufficiently to characterize the difference between futures and forwards as driven by the covariance between LIBOR and futures prices.

Appendix B

Here we show that the convexity bias is created by a difference in the manner in which interest is calculated for the spot and futures instruments. Consider a party at time 0 needing to borrow \$1 at time t, with interest and principal to be repaid at $t + \tau$. Fearing an increase in interest rates, the party hedges by selling N interest rate futures contracts at the price $f_0(t)$, which expires at time t. The underlying interest rate is LIBOR, which takes on a value of L_t at time t. The futures contract is *not* settled until expiration. We ignore day-count adjustments such as 90/360, as these factors do not affect our conclusions. The futures price at expiration is $f_t(t)$, which will either be $1 - L_t$ if the futures contract is structured (like the actual Eurodollar contract) as a discount instrument or $1/(1 + L_t)$ if the futures contract is structured as an add-on instrument (like actual spot Eurodollars). At time t, the futures contract will be cash settled for $-N(f_f(t) - f_0(t))$. The hedger needs \$1 at time t so it will borrow the amount B such that

$$- N(f_t(t) - f_0(t)) + B = 1.$$

Solving for B gives

$$B = 1 + N(f_t(t) - f_0(t)).$$

The exact form of B will depend on how the futures contract is structured, as either a discount instrument or an add-on instrument. The amount borrowed to be paid back at $t + \tau$ will be B, compounded in one of the following two ways:

$B(1 + L_t)$ if the loan is an add-on loan, or

$B/(1 - L_t)$ if the loan is a discount loan.

If the hedge is successful the amount paid back at $t + \tau$ will not contain the term L_t. Thus, the transaction would be risk-free, and the futures contract would be priced by the well-known cost-of-carry formula, which is the same formula that gives the forward price. Thus, forward and futures prices would be equivalent.

Proposition 1: *If the loan pays interest in the add-on manner and the futures contract is designed as a discount instrument, a perfect hedge is not possible.*

Proof: The futures price at expiration is $f_t(t) = 1 - L_t$. The amount borrowed is

$$B = 1 + N(1 - L_t - f_0(t)).$$

The amount paid back at $t + \tau$ will be

$$(1 + N(1 - L_t - f_0(t)))(1 + L_t)$$
$$= (1 + L_t) + N(1 - L_t)(1 + L_t) - Nf_0(t)(1 + L_t).$$

Since $f_0(t)$ is known at time 0, we could have set N to $1/f_0(t)$, which would eliminate two of the L_t terms above but leave $(1/f_0(t))(1 - L_t)(1 + L_t)$, which cannot be eliminated. Thus, this hedge is not perfect.

Proposition 2: *If the loan pays interest in the discount manner and the futures contract is designed as a discount instrument, a perfect hedge is possible.*

Proof: The amount borrowed is the same as in Proposition 1. The amount paid back at $t + \tau$ will be

$$(1 + N(1 - L_t - f_0(t)))/(1 - L_t)$$
$$= 1/(1 - L_t) + N(1 - L_t)/(1 - L_t) - Nf_0(t)/(1 - L_t).$$

We could have set N to $1/f_0(t)$, leaving a loan payoff of

$$1/(f_0(t),$$

which contains no terms related to L_t. Thus, this case is a perfect hedge.

Proposition 3: *If the loan pays interest in the add-on manner and the futures contract is designed as an add-on instrument, a perfect hedge is possible.*

Proof: The amount borrowed is

$$B = 1 + N((1/(1 + L_t)) - f_0(t)).$$

The amount paid back at $t + \tau$ is

$$(1 + N((1/(1 + L_t)) - f_0(t)))(1 + L_t)$$
$$= (1 + L_t) + N - Nf_0(t)(1 + L_t).$$

Setting N to $1/f_0(t)$ results in cancellation of the terms containing L_t, leaving a loan payoff of $1/f_0(t)$. Thus, the transaction is a perfect hedge.

Proposition 4. *If the loan pays interest in the discount manner and the futures contract is designed as an add-on instrument, a perfect hedge is not possible.*

Proof: The amount borrowed is the same as in Proposition 3. The amount paid back at $t + \tau$ is

$$(1 + N((1/(1 + L_t)) - f_0(t)))/(1 - L_t)$$
$$= 1/(1 - L_t) + N(1 - L_t)/(1 + L_t) - Nf_0(t)/(1 - L_t).$$

If we set N to $1/f_0(t)$, then we eliminate some of the terms containing L_t, but we cannot eliminate all of them. Thus, this transaction is not a perfect hedge.

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